CSWIM Workshop. ORNL, October 2007.

FLUID-KINETIC FORMULATION WITH EC-RF SOURCES FOR SLOW-MHD STUDIES

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With acknowledgments to C.C. Hegna and J.D. Callen

I. GENERAL CONSIDERATIONS

The proposed RF/MHD approach is based on an underlying low-frequency kinetic equation, obtained after averaging over the high-frequency RF fields, of the form:

$$\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial t} \;+\; \mathbf{v}\cdot\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{x}} \;+\; \frac{e}{m}\Big(\mathbf{E}+\mathbf{v}\times\mathbf{B}\Big)\cdot\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{v}} \;=\; \sum_s C(f,f_s) \;+\; Q^{RF}(f),$$

with the Fokker-Plank collision operator:

$$C(f,f_s) \;=\; -\,\frac{c^4 e^2 e_s^2 \ln\Lambda_s}{8\pi m}\,\frac{\partial}{\partial \mathbf{v}}\cdot\int d^3\mathbf{w}\;\mathsf{\mathbf{U}}(\mathbf{v},\mathbf{w})\cdot\Big[\frac{f(\mathbf{v},\mathbf{x},t)}{m_s}\,\frac{\partial f_s(\mathbf{w},\mathbf{x},t)}{\partial \mathbf{w}} \;-\; \frac{f_s(\mathbf{w},\mathbf{x},t)}{m}\,\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{v}}\Big]\,,$$

and the quasi-linear RF velocity-space diffusion operator:

$$Q^{RF}(f) \;=\; \frac{\partial}{\partial \mathbf{v}}\cdot\mathsf{\mathbf{D}}^{RF}(\mathbf{v},\mathbf{x},t)\cdot\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{v}}\;.$$

THE AVAILABLE STANDARD FORM OF THE Q^{RF} OPERATOR IS DERIVED FOR A HOMOGENEOUS BACKGROUND MAGNETIC FIELD AND VANISHING BACKGROUND ELECTRIC FIELD.

IN ORDER TO IGNORE CORRECTIONS DUE TO THE MAGNETIC INHOMOGENEITY, CONSIDER ONLY ELECTRON-CYCLOTRON RF WAVES WHOSE INTERACTION WITH THE PLASMA COULD BE TREATED IN A LOCAL APPROXIMATION.

FOR THE PROPAGATION OF THESE EC WAVES, RAY TRACING SHOULD BE SUFFICIENT.

IT IS NOT CLEAR WHAT THE CONSEQUENCES OF IGNORING THE BACKGROUND ELECTRIC FIELD IN Q^{RF} ARE.

II. ELECTRON FLUID EQUATIONS WITH RF SOURCE TERMS

Taking velocity moments of the electron kinetic equation, RF source terms appear in the generalized Ohm's law, the electron mean pressure equation and the electron pressure anisotropy (sometimes called parallel viscosity) equation:

$$\mathbf{E} \ + \ \mathbf{u}_e \times \mathbf{B} \ + \ \frac{1}{en}\Big\{\nabla \cdot \Big[p_e \mathbf{I} + (p_{e\|} - p_{e\perp})(\mathbf{bb} - \mathbf{I}/3)\Big] \ - \ \mathbf{F}_e^{coll} \ - \ \mathbf{F}_e^{RF}\Big\} \ = \ 0,$$

$$\frac{3}{2}\Big[\frac{\partial p_e}{\partial t} + \nabla \cdot \Big(p_e \mathbf{u}_e\Big)\Big] \ + \ p_e \nabla \cdot \mathbf{u}_e \ + \ (p_{e\|} - p_{e\perp})\Big\{\mathbf{b} \cdot \Big[(\mathbf{b} \cdot \nabla)\mathbf{u}_e\Big] - \nabla \cdot \mathbf{u}_e/3\Big\} +$$

$$+ \nabla \cdot \Big(q_{e\|}\mathbf{b} + \mathbf{q}_{e\perp}\Big) \ - \ g_e^{coll} \ - \ g_e^{RF} \ = \ 0 \ ,$$

$$\frac{\partial (p_{e\|} - p_{e\perp})}{\partial t} \ + \ \nabla \cdot \Big[(p_{e\|} - p_{e\perp})\mathbf{u}_e\Big] \ + \ (p_{e\|} - p_{e\perp})\Big\{\mathbf{b} \cdot \Big[(\mathbf{b} \cdot \nabla)\mathbf{u}_e\Big] + \nabla \cdot \mathbf{u}_e/3\Big\} +$$

$$+ p_e\Big\{\mathbf{b} \cdot \Big[3(\mathbf{b} \cdot \nabla)\mathbf{u}_e\Big] - \nabla \cdot \mathbf{u}_e\Big\} \ + \ \nabla \cdot \Big[(3q_{eB\|} - q_{e\|})\mathbf{b} + 3\mathbf{q}_{eB\perp} - \mathbf{q}_{e\perp}\Big] +$$

$$+ 3(q_{e\|} - q_{eB\|})\mathbf{b} \cdot \nabla(\ln \ B) \ - \ 6\mathbf{q}_{eB\perp} \cdot \boldsymbol{\kappa} \ + \ g_e^{coll} \ + \ g_e^{RF} \ - \ 3g_{eB}^{coll} \ - \ 3g_{eB}^{RF} \ = \ 0 \ .$$

The RF source terms in the electron equations are:

$$\mathbf{F}_e^{RF} \;=\; m_e \int d^3\mathbf{v}\;(\mathbf{v}-\mathbf{u}_e)\;Q^{RF}(f_e),$$

$$g_e^{RF} \;=\; (m_e/2) \int d^3\mathbf{v}\;|\mathbf{v}-\mathbf{u}_e|^2\;Q^{RF}(f_e),$$

$$g_{eB}^{RF} \;=\; (m_e/2) \int d^3\mathbf{v}\;[(\mathbf{v}-\mathbf{u}_e)\cdot\mathbf{b}]^2\;Q^{RF}(f_e)$$

For the slow instabilities of interest, the electron distribution function can be assumed to be close to a Maxwellian. Since the amplitude of the RF source terms is also small, these can be approximated by the corresponding explicit representations obtained with a Maxwellian distribution $f_e = f_{eM}$.

In the collisionality regimes of interest, the collisional (diffusive) parts of the perpendicular heat fluxes should be negligible. The corresponding collision-independent (diamagnetic) parts are also completely specified within the required accuracy if the lowest-order electron distribution function is (two-temperature) Maxwellian:

$$\mathbf{q}_{e\perp} \ = \ - \frac{1}{eB}\mathbf{b} \times \Big[p_{e\perp}\nabla\Big(\frac{p_{e\parallel} + 4p_{e\perp}}{2n}\Big) + \frac{p_{e\parallel}(p_{e\parallel} - p_{e\perp})}{n}\boldsymbol{\kappa}\Big] \ ,$$

$$\mathbf{q}_{eB\perp} \ = \ - \frac{1}{eB}\mathbf{b} \times \Big[p_{e\perp}\nabla\Big(\frac{p_{e\parallel}}{2n}\Big) + \frac{p_{e\parallel}(p_{e\parallel} - p_{e\perp)}}{n}\boldsymbol{\kappa}\Big] \ .$$

The electron closure terms that must be provided by kinetic theory are:

The two independent parallel heat fluxes:

$$q_{e\parallel} \ = \ (m_e/2) \int d^3\mathbf{v} \ [(\mathbf{v}-\mathbf{u}_e)\cdot\mathbf{b}] \ |\mathbf{v}-\mathbf{u}_e|^2 \ f_e \ ,$$

$$q_{eB\parallel} \ = \ (m_e/2) \int d^3\mathbf{v} \ [(\mathbf{v}-\mathbf{u}_e)\cdot\mathbf{b}]^3 \ f_e \ .$$

The collisional moments:

$$\mathbf{F}_e^{coll} \ = \ m_e \int d^3\mathbf{v} \ (\mathbf{v}-\mathbf{u}_e) \ \sum_s C(f_e, f_s),$$

$$g_e^{coll} \ = \ (m_e/2) \int d^3\mathbf{v} \ |\mathbf{v}-\mathbf{u}_e|^2 \ \sum_s C(f_e, f_s),$$

$$g_{eB}^{coll} \ = \ (m_e/2) \int d^3\mathbf{v} \ [(\mathbf{v}-\mathbf{u}_e)\cdot\mathbf{b}]^2 \ \sum_s C(f_e, f_s)$$

III. DRIFT-KINETIC EVALUATION OF THE FLUID CLOSURES

The electron fluid closure terms can be evaluated from the solution of a gyrophase-averaged drift-kinetic equation. Such equation should fulfill the following requirements:

Three spatial dimensions in order to analyze non-axisymetric, non-linear magnetic island evolution.

Two velocity dimensions. Magnetic moment no longer conserved when RF and FLR effects are included.

Velocity moments in agreement with the macroscopic fluid equations. Include terms beyond the lowest (zero-Larmor-radius) order, that are inversely proportional to eB but independent of the mass.

A FINITE-LARMOR-RADIUS FORM OF THE DRIFT-KINETIC EQUATION HAS BEEN DERIVED, HAVING THE FOLLOWING DESIRABLE FEATURES:

Accurate to the first significant finite-Larmor-radius order and valid for arbitrary macroscopic flows.

Use of the full macroscopic flow velocity, u, to define the moving frame. Electric field exactly eliminated algebraically and no reference to the $\mathbf{E} \times \mathbf{B}$ or any other drifts.

Formulation in terms of the standard MHD variables (macroscopic flow velocity and magnetic field) only. This should facilitate the coupling to M3D, NIMROD or other MHD-like codes.

Velocity moments reproduce all the previously derived fluid results, including the higher-moment and higher-order in the Larmor radius results.

In its full generality, this "driftless" FLR drift-kinetic equation is (for any species and dropping the species index):

$$\frac{\partial \bar{f}(v'_\parallel, v'_\perp, \mathbf{x}, t)}{\partial t} \ + \ \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} \ + \ \dot{v}'_\parallel \frac{\partial \bar{f}}{\partial v'_\parallel} \ + \ \dot{v}'_\perp \frac{\partial \bar{f}}{\partial v'_\perp} \ = \ \bar{C} \ + \ \bar{Q}^{RF}(f_M) \ ,$$

where the relative velocity variable is $\mathbf{v}' = \mathbf{v} - \mathbf{u}(\mathbf{x}, t)$, with $\mathbf{u}(\mathbf{x}, t)$ the macroscopic flow velocity, and the overbars indicate gyrophase averaging.

The coefficient functions are:

$$\dot{\mathbf{x}} \ = \ \mathbf{u} + v'_\parallel \mathbf{b} \ + \ \frac{v'^2_\perp}{2} \nabla \times \frac{\mathbf{b}}{\Omega_c} \ + \ \frac{\mathbf{b}}{\Omega_c} \times \Big[\frac{\mathbf{F}}{mn} + 2 v'_\parallel (\mathbf{b} \cdot \nabla) \mathbf{u} + \Big(v'^2_\parallel - \frac{v'^2_\perp}{2} \Big) \boldsymbol{\kappa} \Big]$$

with

$$\mathbf{F}(\mathbf{x}, t) \ = \ - \ en(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \ + \ mn \Big[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \Big] \ = \ - \ \nabla \cdot \mathsf{P} \ + \ \mathbf{F}^{coll} \ ,$$

$$\dot{v}'_\parallel \ = \ - \ \frac{\mathbf{b}\cdot\mathbf{F}}{mn} \ - \ v'_\parallel \mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \ - \ \frac{v'^2_\perp}{2}\mathbf{b}\cdot\nabla\ln B \ + \ \frac{v'^2_\perp}{2}\nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b}+v'_\parallel\boldsymbol{\kappa})\Big] -$$

$$- \Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b}+v'_\parallel\boldsymbol{\kappa})\Big]\cdot\Big[\frac{\mathbf{F}}{mn}+2v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u}+v'^2_\parallel\boldsymbol{\kappa}\Big] \ - \ \frac{v'^2_\perp}{2}\sigma$$

with

$$\boldsymbol{\omega}(\mathbf{x},t) \ = \ \nabla\times\mathbf{u} \qquad \textit{and} \qquad \sigma(\mathbf{x},t) \ = \ \frac{1}{4\Omega_c}\epsilon_{jkl}b_j\Big(\frac{\partial b_k}{\partial x_m}+\frac{\partial b_m}{\partial x_k}\Big)(\delta_{mn}-b_m b_n)\Big(\frac{\partial u_l}{\partial x_n}+\frac{\partial u_n}{\partial x_l}\Big) \ ,$$

and

$$\dot{v}'_\perp \ = \ \frac{v'_\perp}{2}\Big\{\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \ - \nabla\cdot\mathbf{u} \ + \ v'_\parallel\mathbf{b}\cdot\nabla\ln B \ - \ \nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times\Big(\frac{\mathbf{F}}{mn}+2v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u}+v'^2_\parallel\boldsymbol{\kappa}\Big)\Big] +$$

$$+ \ 2\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big]\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}+v'_\parallel\boldsymbol{\kappa}\Big] \ + \ \Big(\frac{\mathbf{b}}{\Omega_c}\times\boldsymbol{\kappa}\Big)\cdot\Big[\frac{\mathbf{F}}{mn}+4v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u}\Big]\Big\} \ .$$

FOR SMALL-MASS ELECTRONS, SLOW TIME EVOLUTION ($\partial/\partial t < \omega_*$), SLOW FLOWS ($u < u_*$), AND USING AS PHASE-SPACE VARIABLES THE KINETIC ENERGY $\varepsilon = m(v_\|'^2 + v_\perp'^2)/2$ AND THE MAGNETIC MOMENT $\mu = mv_\perp'^2/(2B)$:

$$\dot{\mathbf{x}} \cdot \frac{\partial \bar{f}(\varepsilon,\mu,\mathbf{x},t)}{\partial \mathbf{x}} \;+\; \dot{\varepsilon}\,\frac{\partial \bar{f}}{\partial \varepsilon} \;+\; \dot{\mu}\,\frac{\partial \bar{f}}{\partial \mu} \;=\; \bar{C} \;+\; \bar{Q}^{RF}(f_M)\;,$$

where

$$\dot{\mathbf{x}} \;=\; \Big[2(\varepsilon - \mu B)/m\Big]^{1/2}\mathbf{b} \;+\; \mu B\;\nabla\times\frac{\mathbf{b}}{m\Omega_c} \;+\; \frac{\mathbf{b}}{m\Omega_c}\times\Big[\frac{\mathbf{F}}{n} + (2\varepsilon - 3\mu B)\boldsymbol{\kappa}\Big]\;,$$

$$\dot{\varepsilon} \;=\; -\,\Big[2(\varepsilon - \mu B)/m\Big]^{1/2}\frac{\mathbf{b}\cdot\mathbf{F}}{n} \;-\; \mu B\;\nabla\cdot\Big(\frac{\mathbf{b}}{m\Omega_c}\times\frac{\mathbf{F}}{n}\Big) \;+\; (2\varepsilon - 3\mu B)\Big(\frac{\mathbf{b}}{m\Omega_c}\times\frac{\mathbf{F}}{n}\Big)\cdot\boldsymbol{\kappa}\;,$$

$$\dot{\mu} \;=\; \frac{\mu}{m\Omega_c}\mathbf{b}\cdot\Big\{\nabla\times\Big[2(\varepsilon - \mu B)\boldsymbol{\kappa}\Big] \;-\; [\mathbf{b}\cdot(\nabla\times\mathbf{b})]\;\Big(\frac{\mathbf{F}}{n} + \mu\nabla B\Big)\Big\}\;.$$

IV. ACTION ITEM

IN ORDER TO PROCEED IN A COHERENT FASHION, IT WOULD BE HIGHLY DESIRABLE TO REACH A CONSENSUS ON THE APPROPRIATE RELATIVE ORDERINGS AMONG THE FOLLOWING DIMENSIONLESS RATIOS:

$$m_e/m_i \;\; , \;\; \rho_i/L \;\; , \;\; (\partial/\partial t)/\Omega_{ci} \;\; , \;\; \nu_i/\Omega_{ci} \;\; , \;\; u_s/v_{thi} \;\; , \;\; (p_{s\|} - p_{s\perp})/p_s \;\; , \;\; (f_s - f_{sM})/f_{sM}$$

University of Wisconsin, Madison, December 2007.

KINETIC THEORY OF A MAGNETIZED PLASMA IN THE MACROSCOPIC FLOW REFERENCE FRAME*

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*Work supported by the U.S. Department of Energy

MOTIVATION: PARALLEL CLOSURES OF THE FLUID MOMENT EQUATIONS, i.e. EVALUATION OF THE CHEW-GOLDBERGER-LOW (GYROTROPIC) PARTS OF THE STRESS AND STRESS-FLUX TENSORS:

$$\mathsf{P}_{jk}^{CGL} = p_\perp \delta_{jk} + (p_\parallel - p_\perp) b_j b_k = p\,\delta_{jk} + (p_\parallel - p_\perp)(b_j b_k - \delta_{jk}/3),$$

$$\textbf{with} \qquad p = (2p_\perp + p_\parallel)/3\ .$$

and

$$\mathsf{Q}_{jkl}^{CGL} = q_{T\parallel}(\delta_{jk} b_l + \delta_{kl} b_j + \delta_{lj} b_k) + (2q_{B\parallel} - 3q_{T\parallel}) b_j b_k b_l\ ,$$

$$\textbf{with} \qquad q_\parallel = q_{B\parallel} + q_{T\parallel}\ .$$

(Species indices will be omitted when equations apply to either species or to the ions. The electron species index will be made explicit in the simplified form of the electron equations after taking the small electron mass limit.)

THE CGL VARIABLES ARE MOMENTS OF THE GYROPHASE-AVERAGED PART OF THE DISTRIBUTION FUNCTION IN THE MEAN FLOW REFERENCE FRAME:

$$\begin{aligned} p &= (m/3) \int d^3\mathbf{v} \; |\mathbf{v}-\mathbf{u}|^2 \; \bar{f} \; , \\ p_\parallel - p_\perp &= (m/2) \int d^3\mathbf{v} \; \left\{ 3[(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2 - |\mathbf{v}-\mathbf{u}|^2 \right\} \; \bar{f} \; , \end{aligned}$$

$$\begin{aligned} q_\parallel &= (m/2) \int d^3\mathbf{v} \; [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}] \; |\mathbf{v}-\mathbf{u}|^2 \; \bar{f} \; , \\ q_{B\parallel} &= (m/2) \int d^3\mathbf{v} \; [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^3 \; \bar{f} \; . \end{aligned}$$

ONE MAY CHOOSE TO DETERMINE SOME OF THESE VARIABLES FROM THEIR (NOT CLOSED) FLUID MOMENT EVOLUTION EQUATIONS AND THE REST (THE SO-CALLED CLOSURE TERMS) FROM THE SOLUTION OF A DRIFT-KINETIC EQUATION FOR $\bar{f}$. SUCH DRIFT-KINETIC EQUATION SHOULD BE COMPATIBLE WITH THE FLUID MOMENT EQUATIONS BEING USED.

KEEPING $O(pv_{th\iota}/L) + O(\delta p v_{th\iota}/L)$ IN A FINITE-LARMOR-RADIUS, FAST DYNAMICS ORDERING [i.e. $\partial/\partial t = O(\delta\Omega_{c\iota}) + O(\delta^2\Omega_{c\iota})$ and $u = O(v_{th\iota}) + O(\delta v_{th\iota})$ with $\delta \sim \rho_\iota/L$], THE FLUID EVOLUTION EQUATIONS FOR THE COMPONENTS OF $\boldsymbol{\mathsf{P}}^{CGL}$ ARE:

$$\frac{3}{2}\Big[\frac{\partial p}{\partial t} + \nabla\cdot\big(p\mathbf{u}\big)\Big] \; + \; p\nabla\cdot\mathbf{u} \; + \; (p_\parallel - p_\perp)\Big\{\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}\big] - \nabla\cdot\mathbf{u}/3\Big\} \; + \; \nabla\cdot\big(q_\parallel\mathbf{b}\big) \; +$$

$$+ \; \boldsymbol{\mathsf{P}}^{gyr} : (\nabla\mathbf{u}) \; + \; \nabla\cdot\mathbf{q}_\perp \; - g^{coll} \; = \; 0$$

and

$$\frac{\partial(p_\parallel - p_\perp)}{\partial t} \; + \; \nabla\cdot\big[(p_\parallel - p_\perp)\mathbf{u}\big] \; + \; (p_\parallel - p_\perp)\Big\{\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}\big] + \nabla\cdot\mathbf{u}/3\Big\} \; +$$

$$+ \; p\Big\{3\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}\big] - \nabla\cdot\mathbf{u}\Big\} \; + \; \nabla\cdot\big[(3q_{B\parallel} - q_\parallel)\mathbf{b}\big] \; + \; 3(q_\parallel - q_{B\parallel})\mathbf{b}\cdot\nabla(\ln\; B) \; +$$

$$+3\mathbf{b}\cdot\boldsymbol{\mathsf{P}}^{gyr}\cdot(\mathbf{b}\times\boldsymbol{\omega}) - \boldsymbol{\mathsf{P}}^{gyr} : (\nabla\mathbf{u}) + \nabla\cdot(3\mathbf{q}_{B\perp} - \mathbf{q}_\perp) - 6\mathbf{q}_{B\perp}\cdot\boldsymbol{\kappa} + 3(q_\parallel - q_{B\parallel})\sigma + g^{coll} - 3g_B^{coll} \; = \; 0 \, .$$

Here, the non-gyrotropic, FLR terms $\boldsymbol{\mathsf{P}}^{gyr}$, $\mathbf{q}_\perp$, $\mathbf{q}_{B\perp}$ and σ are known (except for a non-Maxwellian contribution to $\mathbf{q}_\perp$ and $\mathbf{q}_{B\perp}$). For the electrons, $\boldsymbol{\mathsf{P}}^{gyr}$ and σ can be neglected.

Specifically,

$$\mathsf{P}^{gyr} : (\nabla \mathbf{u}) \;=\; \mathbf{b} \cdot \mathsf{P}^{gyr} \cdot [2(\mathbf{b} \cdot \nabla)\mathbf{u} + \mathbf{b} \times \boldsymbol{\omega}] \;+\; (q_\parallel - q_{B\parallel})\sigma \;,$$

$$\mathbf{b} \cdot \mathsf{P}^{gyr} \;=\; \frac{m}{eB}\, \mathbf{b} \times [2p_\parallel(\mathbf{b} \cdot \nabla)\mathbf{u} + p_\perp \mathbf{b} \times \boldsymbol{\omega} + \nabla(q_\parallel - q_{B\parallel}) + 2(2q_{B\parallel} - q_\parallel)\boldsymbol{\kappa}] \;,$$

$$\sigma \;=\; \frac{m}{4eB}\epsilon_{jkl} b_j \Big(\frac{\partial b_k}{\partial x_m} + \frac{\partial b_m}{\partial x_k}\Big)(\delta_{mn} - b_m b_n)\Big(\frac{\partial u_l}{\partial x_n} + \frac{\partial u_n}{\partial x_l}\Big) \;,$$

$$\mathbf{q}_\perp \;=\; \frac{1}{eB}\mathbf{b} \times \Big[p_\perp \nabla\Big(\frac{p_\parallel + 4p_\perp}{2n}\Big) + \frac{p_\parallel(p_\parallel - p_\perp)}{n}\boldsymbol{\kappa} + 2m(2q_\parallel - q_{B\parallel})(\mathbf{b} \cdot \nabla)\mathbf{u} + m(q_\parallel - q_{B\parallel})\mathbf{b} \times \boldsymbol{\omega}\Big] \;+\; \tilde{\mathbf{q}}_\perp \;,$$

$$\mathbf{q}_{B\perp} \;=\; \frac{1}{eB}\mathbf{b} \times \Big[p_\perp \nabla\Big(\frac{p_\parallel}{2n}\Big) + \frac{p_\parallel(p_\parallel - p_\perp)}{n}\boldsymbol{\kappa} + 2mq_{B\parallel}(\mathbf{b} \cdot \nabla)\mathbf{u} + m(q_\parallel - q_{B\parallel})\mathbf{b} \times \boldsymbol{\omega}\Big] \;+\; \tilde{\mathbf{q}}_{B\perp} \;,$$

where the $m \rightarrow 0$ limit can be taken for the electrons.

The non-Maxwellian contributions $\tilde{\mathbf{q}}_\perp$ and $\tilde{\mathbf{q}}_{B\perp}$ to the perpendicular heat fluxes are:

$$\tilde{\mathbf{q}}_\perp = \frac{1}{eB}\mathbf{b} \times \left[\nabla \tilde{r}_\perp^{(0)} + (\tilde{r}_\parallel^{(0)} - \tilde{r}_\perp^{(0)})\boldsymbol{\kappa}\right]$$

and

$$\tilde{\mathbf{q}}_{B\perp} = \frac{1}{eB}\mathbf{b} \times \left[\nabla \tilde{r}_{B\perp}^{(0)} + (\tilde{r}_\parallel^{(0)} - 5\tilde{r}_{B\perp}^{(0)})\boldsymbol{\kappa}\right] ,$$

where

$$\tilde{r}_\perp^{(0)} \;=\; (m^2/4) \int d^3\mathbf{v}\; |\mathbf{v}-\mathbf{u}|^2 \left\{ |\mathbf{v}-\mathbf{u}|^2 - [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2 \right\} \left(\bar{f}^{(0)} - f_{2M} \right) ,$$

$$\tilde{r}_\parallel^{(0)} \;=\; (m^2/2) \int d^3\mathbf{v}\; |\mathbf{v}-\mathbf{u}|^2 \; [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2 \left(\bar{f}^{(0)} - f_{2M} \right) ,$$

$$\tilde{r}_{B\perp}^{(0)} \;=\; (m^2/4) \int d^3\mathbf{v}\; [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2 \left\{ |\mathbf{v}-\mathbf{u}|^2 - [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2 \right\} \left(\bar{f}^{(0)} - f_{2M} \right) .$$

DRIFT-KINETIC EVALUATION OF THE FLUID CLOSURES

*** THE VARIABLES NEEDED FOR CLOSURE OF THE FLUID SYSTEM CAN BE DERIVED FROM MOMENTS OF THE GYROPHASE-AVERAGED DISTRIBUTION FUNCTIONS, $\bar{f}$, SATISFYING DRIFT-KINETIC EQUATIONS.**

*** THE VELOCITY MOMENTS OF $\bar{f}$ NEEDED FOR THE FLUID CLOSURE ARE EVALUATED MOST CONVENIENTLY IN THE MOVING FRAME OF THE FULL MACROSCOPIC FLOW VELOCITY, $\mathbf{u}(\mathbf{x},t)$.**

*** FOR SONIC-SCALE DYNAMICS WITH LARGE PERPENDICULAR FLOWS AND ELECTRIC FIELDS ($u_\perp \sim v_{th\iota}$ and $E_\perp \sim v_{th\iota}B$), THE ION DRIFT-KINETIC EQUATION MUST BE DERIVED IN A MOVING FRAME CLOSE TO THE ELECTRIC DRIFT VELOCITY $\mathbf{u}_E(\mathbf{x},t) = \mathbf{E} \times \mathbf{B}/B^2$, SUCH AS $\mathbf{u}_E(\mathbf{x},t)$ ITSELF OR $\mathbf{u}_\iota(\mathbf{x},t)$.**

*** TO DETERMINE THE COLLISIONAL TERMS IN A LOW-COLLISIONALITY REGIME ($\nu_\iota \lesssim \omega_*$) AND THE PERPENDICULAR HEAT FLUX CLOSURE TERMS $\tilde{\mathbf{q}}_{\alpha\perp}$ AND $\tilde{\mathbf{q}}_{\alpha B\perp}$ WITHIN THE REQUIRED ACCURACY, ONLY THE LOWEST-ORDER OR ZERO-LARMOR-RADIUS DISTRIBUTION FUNCTIONS $\bar{f} = \bar{f}^{(0)}$ ARE NEEDED.**

*** TO DETERMINE THE PRESSURES AND THE PARALLEL HEAT FLUXES (OR THE COEFFICIENT FUNCTIONS IN THEIR FLUID EVOLUTION EQUATIONS) WITHIN THE REQUIRED ACCURACY, FIRST-ORDER FLR SOLUTIONS OF THE DRIFT-KINETIC EQUATION, $\bar{f} = \bar{f}^{(0)} + \bar{f}^{(1)}$, ARE NECESSARY.**

THE DRIFT-KINETIC EQUATION FOR THE GYROPHASE-AVERAGED $\bar{f}$ IS OBTAINED BY EXPLOITING THE FACT THAT $(e/m)\ (\mathbf{v} \times \mathbf{B}) \cdot \partial f(\mathbf{v},\mathbf{x},t)/\partial \mathbf{v}$ IS THE DOMINANT TERM IN THE VLASOV-BOLTZMANN EQUATION FOR A MAGNETIZED PLASMA COMPONENT:

$$\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial t} + \mathbf{v} \cdot \frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{x}} + \frac{e}{m}\Big(\mathbf{E} + \mathbf{v} \times \mathbf{B}\Big) \cdot \frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial \mathbf{v}} = \sum_s C(f, f_s) \ .$$

THIS REQUIRES WORKING IN A REFERENCE FRAME WHERE THE ELECTRIC FIELD IS SUFFICIENTLY SMALL ($E \ll v_{th} B$). SINCE $E_{\|}$ IS ALWAYS SMALL, THIS CAN BE ACHIEVED WITH AN APPROPRIATE GALILEAN BOOST .

CONSIDER, FOR SIMPLICITY, THE COLLISIONLESS LIMIT AND TRANSFORM THE VLASOV-BOLTZMANN EQUATION TO A REFERENCE FRAME MOVING WITH VELOCITY $\mathbf{u}(\mathrm{x}, t)$ RELATIVE TO THE LABORATORY:

$$t = t\ , \qquad \mathbf{x} = \mathbf{x}\ , \qquad \mathbf{v}' = \mathbf{v} - \mathbf{u}(\mathbf{x}, t)\ ,$$

$$\frac{\partial f(\mathbf{v}', \mathbf{x}, t)}{\partial t} \ + \ (\mathbf{v}' + \mathbf{u}) \cdot \frac{\partial f(\mathbf{v}', \mathbf{x}, t)}{\partial \mathbf{x}} \ + \ \left[\Omega_c \mathbf{v}' \times \mathbf{b} + \frac{\mathbf{F}}{mn} - (\mathbf{v}' \cdot \nabla)\mathbf{u}\right] \cdot \frac{\partial f(\mathbf{v}', \mathbf{x}, t)}{\partial \mathbf{v}'} \ = \ 0\ ,$$

with

$$\mathbf{F}(\mathbf{x}, t) \ = \ en(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \ - \ mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right]\ .$$

Here $\mathbf{u}(\mathbf{x}, t)$ is arbitrary, provided only $\mathbf{F}/(mn) \ll \Omega_c v_{th}$.

POSSIBLE CHOICES FOR $\mathbf{u}(\mathbf{x},t)$ AND $\mathbf{F}(\mathbf{x},t) = en(\mathbf{E}+\mathbf{u}\times\mathbf{B}) - mn[\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u}]$:

1.) **IF** $E \ll v_{th}B$ **,** $\mathbf{u}=0$ **AND** $\mathbf{F} = en\mathbf{E}$

2.) $\mathbf{u} = \mathbf{u}_E = \mathbf{E}\times\mathbf{B}/B^2$ **AND** $\mathbf{F} = enE_{\|}\mathbf{b} - mn[\partial\mathbf{u}_E/\partial t + (\mathbf{u}_E\cdot\nabla)\mathbf{u}_E]$

3.) $\mathbf{u} = n^{-1}\int d^3\mathbf{v}\;\mathbf{v}\;f$ **AND** $\mathbf{F} = \nabla\cdot\mathbf{P} = \nabla\cdot(\mathbf{P}^{CGL} + \mathbf{P}^{gyr})$

CHANGE VARIABLES TO A CYLINDRICAL COORDINATE SYSTEM IN VELOCITY SPACE, LOCALLY ALIGNED WITH THE MAGNETIC FIELD:

$$t = t\ , \qquad \mathbf{x} = \mathbf{x}\ , \qquad \mathbf{v}' = v'_\parallel\ \mathbf{b}(\mathbf{x},t)\ +\ v'_\perp\ [\cos\alpha\ \mathbf{e}_1(\mathbf{x},t) + \sin\alpha\ \mathbf{e}_2(\mathbf{x},t)]\ ,$$

$$\Omega_c\ \frac{\partial f(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t)}{\partial\alpha} = \sum_{l=-2}^{2} e^{il\alpha}\left[\Lambda_l\ f\ +\ \lambda_l\ \frac{\partial f}{\partial\alpha}\right]\ ,$$

WHERE $\Lambda_l(\partial/\partial v'_\parallel, \partial/\partial v'_\perp, \partial/\partial\mathbf{x}, \partial/\partial t, v'_\parallel, v'_\perp, \mathbf{x}, t) = \Lambda^*_{-l}$ **ARE GYROPHASE-INDEPENDENT OPERATORS AND** $\lambda_l(v'_\parallel, v'_\perp, \mathbf{x}, t) = \lambda^*_{-l}$ **ARE GYROPHASE-INDEPENDENT FUNCTIONS.**

Specifically,

$$\Lambda_0 \;=\; \frac{\partial}{\partial t} \;+\; (\mathbf{u}+v'_\parallel\mathbf{b})\cdot\frac{\partial}{\partial \mathbf{x}} \;+\; \Big\{\frac{\mathbf{b}\cdot\mathbf{F}}{mn} - v'_\parallel\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] + \frac{v'^2_\perp}{2}\nabla\cdot\mathbf{b}\Big\}\,\frac{\partial}{\partial v'_\parallel} \;+$$

$$+\;\frac{v'_\perp}{2}\Big\{\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] - \nabla\cdot\mathbf{u} - v'_\parallel\nabla\cdot\mathbf{b}\Big\}\,\frac{\partial}{\partial v'_\perp}\;,$$

$$\Lambda_1 \;=\; \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big[\frac{\partial\mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \mathbf{b}\times\boldsymbol{\omega} + v'_\parallel\boldsymbol{\kappa}\Big]\,\frac{\partial}{\partial v'_\parallel} \;+$$

$$+\;\frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big\{\frac{\mathbf{F}}{mn} - v'_\parallel\,\Big[\frac{\partial\mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u} + v'_\parallel\boldsymbol{\kappa}\Big]\Big\}\,\frac{\partial}{\partial v'_\perp}\;,$$

$$\Lambda_2 \;=\; \frac{iv'^2_\perp}{4}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big[\nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)\Big]\,\frac{\partial}{\partial v'_\parallel} \;-$$

$$-\;\frac{v'_\perp}{4}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big\{[(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\nabla]\mathbf{u} + iv'_\parallel\;\nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)\Big\}\,\frac{\partial}{\partial v'_\perp}\;,$$

$$\lambda_0 \;=\; \frac{1}{2}\Big\{\mathbf{e}_1\cdot\Big[\frac{\partial \mathbf{e}_2}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{e}_2+(\mathbf{e}_2\cdot\nabla)\mathbf{u}+v'_{\|}(\mathbf{b}\cdot\nabla)\mathbf{e}_2\Big]-$$

$$-\;\mathbf{e}_2\cdot\Big[\frac{\partial \mathbf{e}_1}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{e}_1+(\mathbf{e}_1\cdot\nabla)\mathbf{u}+v'_{\|}(\mathbf{b}\cdot\nabla)\mathbf{e}_1\Big]\;-\;v'_{\|}\mathbf{b}\cdot(\nabla\times\mathbf{b})\Big\}\;,$$

$$\lambda_1 \;=\; \frac{i}{2v'_{\perp}}(\mathbf{e}_1-i\mathbf{e}_2)\cdot\Big\{\frac{\mathbf{F}}{mn}-v'_{\|}\;\Big[\frac{\partial \mathbf{b}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{b}+(\mathbf{b}\cdot\nabla)\mathbf{u}+v'_{\|}\boldsymbol{\kappa}\Big]\Big\}\;-\;\frac{v'_{\perp}}{2}\mathbf{b}\cdot\Big[\nabla\times(\mathbf{e}_1-i\mathbf{e}_2)\Big]\;,$$

$$\lambda_2 \;=\; -\frac{i}{4}(\mathbf{e}_1-i\mathbf{e}_2)\cdot\Big\{[(\mathbf{e}_1-i\mathbf{e}_2)\cdot\nabla]\mathbf{u}+iv'_{\|}\;\nabla\times(\mathbf{e}_1-i\mathbf{e}_2)\Big\}\;.$$

Using Faraday's law, with the definition of F to solve for the electric field:

$$\Lambda_1 \;=\; \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \frac{v'_\perp}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\mathbf{Z}\,\frac{\partial}{\partial v'_\parallel} \;+\; \frac{1}{2}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\,\mathbf{Y}\Big)\,\frac{\partial}{\partial v'_\perp}$$

and

$$\lambda_1 \;=\; \frac{i}{2v'_\perp}(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big) \;-\; \frac{v'_\perp}{2}\mathbf{b}\cdot\Big[\nabla\times(\mathbf{e}_1 - i\mathbf{e}_2)\Big]\,,$$

where

$$\mathbf{Z}(v'_\parallel,\mathbf{x},t) \;=\; -\mathbf{b}\times\boldsymbol{\omega} + v'_\parallel\boldsymbol{\kappa} - \frac{1}{\Omega_c}\,\nabla\times\Big[\frac{\mathbf{F}}{mn} + \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big]$$

and

$$\mathbf{Y}(v'_\parallel,\mathbf{x},t) \;=\; 2(\mathbf{b}\cdot\nabla)\mathbf{u} + v'_\parallel\boldsymbol{\kappa} - \frac{1}{\Omega_c}\,\nabla\times\Big[\frac{\mathbf{F}}{mn} + \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\Big]$$

FOURIER SERIES SOLUTION IN HARMONICS OF THE GYROFREQUENCY:

$$\Omega_c \, \frac{\partial f(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t)}{\partial \alpha} = \sum_{l=-2}^{2} e^{il\alpha} \left[\Lambda_l \, f \; + \; \lambda_l \, \frac{\partial f}{\partial \alpha} \right] ,$$

$$f(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t) \; = \sum_{l=-\infty}^{\infty} e^{il\alpha} \, f_l(v'_\parallel, v'_\perp, \mathbf{x}, t) \qquad [f_0 \equiv \bar{f}] \; ,$$

$$f_l \; = \; \frac{1}{il\Omega_c} \sum_{l'=-2}^{2} \Big[\Lambda_{l'} f_{l-l'} \; + \; i(l-l')\lambda_{l'} f_{l-l'} \Big] \qquad \textbf{for} \qquad l \neq 0$$

and

$$\sum_{l=-2}^{2} \Big(\Lambda_l f_{-l} \; - \; il\lambda_l f_{-l} \Big) \; = \; 0 \; .$$

RECURSIVE SOLUTION FOR STRONG MAGNETIZATION:

$$f_0 = f_0^{(0)} + f_0^{(1)} + \dots \, , \qquad f_{\pm 1} = f_{\pm 1}^{(1)} + \dots \, , \qquad f_{\pm 2} = f_{\pm 2}^{(1)} + \dots \, , \qquad \dots$$

$$\Lambda_0 f_0^{(0)} = 0 \, ,$$

$$f_1^{(1)} = f_{-1}^{(1)*} = \frac{1}{i\Omega_c} \Lambda_1 f_0^{(0)} \, , \qquad f_2^{(1)} = f_{-2}^{(1)*} = \frac{1}{2i\Omega_c} \Lambda_2 f_0^{(0)} \, ,$$

$$(\Lambda_{-2} + 2i\lambda_{-2}) f_2^{(1)} + (\Lambda_{-1} + i\lambda_{-1}) f_1^{(1)} + \Lambda_0 f_0^{(1)} + (\Lambda_1 - i\lambda_1) f_{-1}^{(1)} + (\Lambda_2 - 2i\lambda_2) f_{-2}^{(1)} = 0 \, .$$

Substituting the solutions for $f_{\pm 1}^{(1)}$ and $f_{\pm 2}^{(1)}$:

$$\Lambda_0 (f_0^{(0)} + f_0^{(1)}) \; - \; \frac{1}{i\Omega_c} \Big\{ \big[(\Lambda_1 - i\lambda_1)\Lambda_{-1} - (\Lambda_{-1} + i\lambda_{-1})\Lambda_1 \big] \; + \; \big[(\frac{1}{2}\Lambda_2 - i\lambda_2)\Lambda_{-2} - (\frac{1}{2}\Lambda_{-2} + i\lambda_{-2})\Lambda_2 \big] \; -$$

$$- \, \frac{v'_\perp}{2} \big[(\mathbf{e}_1 - i\mathbf{e}_2) \cdot \nabla \ln B \; \Lambda_{-1} - (\mathbf{e}_1 + i\mathbf{e}_2) \cdot \nabla \ln B \; \Lambda_1 \big] \Big\} f_0^{(0)} \; = \; 0 \, .$$

EVALUATION OF COMMUTATORS:

$$
\begin{aligned}
\frac{1}{i\Omega_c}\Big[(\Lambda_1 - i\lambda_1)\Lambda_{-1} - (\Lambda_{-1} + i\lambda_{-1})\Lambda_1\Big] \;=\; & \frac{1}{\Omega_c}\Big\{\mathbf{b}\times\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big) - \frac{v'^2_\perp}{2}[\mathbf{b}\cdot(\nabla\times\mathbf{b})]\;\mathbf{b}\Big\}\cdot\frac{\partial}{\partial\mathbf{x}} + \\
& + \frac{1}{\Omega_c}\Big\{\Big[\mathbf{b}\times\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big)\Big]\cdot\mathbf{Z} + \frac{v'^2_\perp}{2}\nabla\cdot(\mathbf{Z}\times\mathbf{b})\Big\}\,\frac{\partial}{\partial v'_\parallel} + \\
& \frac{v'_\perp}{2\Omega_c}\Big\{\Big(\mathbf{b}\times\boldsymbol{\kappa}\Big)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big) + \nabla\cdot\Big[\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big)\times\mathbf{b}\Big] + \Big[\mathbf{b}\times\Big(\mathbf{Y} + v'_\parallel\boldsymbol{\kappa}\Big)\Big]\cdot\mathbf{Z}\Big\}\,\frac{\partial}{\partial v'_\perp}\;,
\end{aligned}
$$

$$
\frac{1}{i\Omega_c}\Big[\Big(\frac{1}{2}\Lambda_2 - i\lambda_2\Big)\Lambda_{-2} - \Big(\frac{1}{2}\Lambda_{-2} + i\lambda_{-2}\Big)\Lambda_2\Big] \;=\; \frac{v'^2_\perp}{2}\sigma\,\frac{\partial}{\partial v'_\parallel}\;,
$$

$$
\begin{aligned}
\frac{v'_\perp}{2i\Omega_c}\Big[(\mathbf{e}_1 - i\mathbf{e}_2)\cdot\nabla\ln B\;\Lambda_{-1} - (\mathbf{e}_1 + i\mathbf{e}_2)\cdot\nabla\ln B\;\Lambda_1\Big] \;=\; & \frac{v'^2_\perp}{2\Omega_c}(\mathbf{b}\times\nabla\ln B)\cdot\frac{\partial}{\partial\mathbf{x}} + \\
& + \frac{v'^2_\perp}{2\Omega_c}\Big[(\mathbf{b}\times\nabla\ln B)\cdot\mathbf{Z}\Big]\frac{\partial}{\partial v'_\parallel} \;+\; \frac{v'_\perp}{2\Omega_c}\Big[(\mathbf{b}\times\nabla\ln B)\cdot\Big(\frac{\mathbf{F}}{mn} - v'_\parallel\mathbf{Y}\Big)\Big]\frac{\partial}{\partial v'_\perp}\;,
\end{aligned}
$$

ALL INDEPENDENT OF ($\mathbf{e}_1$, $\mathbf{e}_2$).

FINAL FORM OF THE COLLISIONLESS FLR DRIFT-KINETIC EQUATION:

$$\frac{\partial \bar{f}(v'_\parallel, v'_\perp, \mathbf{x}, t)}{\partial t} \;+\; \dot{\mathbf{x}}\cdot\frac{\partial \bar{f}}{\partial \mathbf{x}} \;+\; \dot{v}'_\parallel\,\frac{\partial \bar{f}}{\partial v'_\parallel} \;+\; \dot{v}'_\perp\,\frac{\partial \bar{f}}{\partial v'_\perp} \;=\; 0\ ,$$

with the coefficient functions:

$$\dot{\mathbf{x}} \;=\; \mathbf{u} + v'_\parallel\mathbf{b} \;+\; \frac{v'^2_\perp}{2}\nabla\times\frac{\mathbf{b}}{\Omega_c} \;-\; \frac{\mathbf{b}}{\Omega_c}\times\Big[\frac{\mathbf{F}}{mn} - 2v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} - \Big(v'^2_\parallel - \frac{v'^2_\perp}{2}\Big)\boldsymbol{\kappa}\Big]\ ,$$

$$\begin{aligned}\dot{v}'_\parallel \;=\; & \frac{\mathbf{b}\cdot\mathbf{F}}{mn} \;-\; v'_\parallel\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \;-\; \frac{v'^2_\perp}{2}\mathbf{b}\cdot\nabla\ln B \;+\; \frac{v'^2_\perp}{2}\nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b} + v'_\parallel\boldsymbol{\kappa})\Big] \;+ \\ & +\;\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b} + v'_\parallel\boldsymbol{\kappa})\Big]\cdot\Big[\frac{\mathbf{F}}{mn} - 2v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} - v'^2_\parallel\boldsymbol{\kappa}\Big] \;-\; \frac{v'^2_\perp}{2}\sigma\ ,\end{aligned}$$

$$\begin{aligned}\dot{v}'_\perp \;=\; & \frac{v'_\perp}{2}\Big\{\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}] \;-\nabla\cdot\mathbf{u} \;+\; v'_\parallel\mathbf{b}\cdot\nabla\ln B \;+\; \nabla\cdot\Big[\frac{\mathbf{b}}{\Omega_c}\times\Big(\frac{\mathbf{F}}{mn} - 2v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} - v'^2_\parallel\boldsymbol{\kappa}\Big)\Big] \;+ \\ & +\;2\Big[\frac{\mathbf{b}}{\Omega_c}\times(\boldsymbol{\omega}\times\mathbf{b})\Big]\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u} + v'_\parallel\boldsymbol{\kappa}\Big] \;-\; \Big(\frac{\mathbf{b}}{\Omega_c}\times\boldsymbol{\kappa}\Big)\cdot\Big[\frac{\mathbf{F}}{mn} - 4v'_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u}\Big]\Big\}\ .\end{aligned}$$

FOR SMALL-MASS ELECTRONS AND USING AS PHASE-SPACE VARIABLES THE MOVING FRAME KINETIC ENERGY $\varepsilon' = m_e(v_\parallel'^2 + v_\perp'^2)/2$ AND MAGNETIC MOMENT $\mu' = m_e v_\perp'^2/(2B)$:

$$\frac{\partial \bar{f}_e(\varepsilon',\mu',\mathbf{x},t)}{\partial t} + \dot{\mathbf{x}}\cdot\frac{\partial \bar{f}_e}{\partial \mathbf{x}} + \dot{\varepsilon}'\,\frac{\partial \bar{f}_e}{\partial \varepsilon'} + \dot{\mu}'\,\frac{\partial \bar{f}_e}{\partial \mu'} = 0\ ,$$

where

$$\dot{\mathbf{x}} = \big[2(\varepsilon' - \mu' B)/m_e\big]^{1/2}\mathbf{b} + \mu' B\,\nabla\times\frac{\mathbf{b}}{m_e\Omega_{ce}} - \frac{\mathbf{b}}{m_e\Omega_{ce}}\times\Big[\frac{\mathbf{F}}{n} - (2\varepsilon' - 3\mu' B)\boldsymbol{\kappa}\Big] + \mathbf{u}\ ,$$

$$\begin{aligned}\dot{\varepsilon}' = {}& \big[2(\varepsilon' - \mu' B)/m_e\big]^{1/2}\,\frac{\mathbf{b}\cdot\mathbf{F}}{n} + \mu' B\,\nabla\cdot\Big(\frac{\mathbf{b}}{m_e\Omega_{ce}}\times\frac{\mathbf{F}}{n}\Big) - \\ & - (2\varepsilon' - 3\mu' B)\Big(\frac{\mathbf{b}}{m_e\Omega_{ce}}\times\frac{\mathbf{F}}{n}\Big)\cdot\boldsymbol{\kappa} - \mu' B\,\nabla\cdot\mathbf{u} - (2\varepsilon' - 3\mu' B)\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}]\ ,\end{aligned}$$

$$\dot{\mu}' = \frac{\mu'}{m_e\Omega_{ce}}\Big\{[\mathbf{b}\cdot(\nabla\times\mathbf{b})]\,\Big[\mathbf{b}\cdot\Big(\frac{\mathbf{F}}{n} - \mu'\nabla B\Big)\Big] + 2(\varepsilon' - \mu' B)\mathbf{b}\cdot(\nabla\times\boldsymbol{\kappa})\Big\}\ .$$

FOR SMALL-MASS ELECTRONS WITH $u_e \sim u_\iota \lesssim v_{th\iota} \ll v_{the}$, THE DRIFT-KINETIC ANALYSIS CAN BE CARRIED OUT IN THE LABORATORY FRAME. THEN, SETTING $\mathbf{u} = 0, \mathbf{F} = -en\mathbf{E}$ AND USING AS PHASE-SPACE VARIABLES THE LABORATORY FRAME $\varepsilon = m(v_\parallel^2 + v_\perp^2)/2$ AND $\mu = mv_\perp^2/(2B)$, ONE GETS:

$$\frac{\partial \bar{f}_e(\varepsilon,\mu,\mathbf{x},t)}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{x}} + \dot{\varepsilon}\, \frac{\partial \bar{f}_e}{\partial \varepsilon} + \dot{\mu}\, \frac{\partial \bar{f}_e}{\partial \mu} = 0 \ ,$$

where

$$\dot{\mathbf{x}} = \Big[2(\varepsilon - \mu B)/m_e\Big]^{1/2}\mathbf{b} + \mu B\, \nabla \times \frac{\mathbf{b}}{m_e\Omega_{ce}} + \frac{\mathbf{b}}{m_e\Omega_{ce}} \times \Big[e\mathbf{E} + (2\varepsilon - 3\mu B)\boldsymbol{\kappa}\Big] \ ,$$

$$\dot{\varepsilon} = -\Big[2(\varepsilon - \mu B)/m_e\Big]^{1/2} e\mathbf{b}\cdot\mathbf{E} + \mu B\, \nabla \cdot \Big(\frac{\mathbf{b}}{B} \times \mathbf{E}\Big) - (2\varepsilon - 3\mu B)\Big(\frac{\mathbf{b}}{B} \times \mathbf{E}\Big) \cdot \boldsymbol{\kappa} \ ,$$

$$\dot{\mu} = \frac{\mu}{m_e\Omega_{ce}}\Big\{ -[\mathbf{b}\cdot(\nabla \times \mathbf{b})]\,\Big[\mathbf{b}\cdot\Big(e\mathbf{E} + \mu\nabla B\Big)\Big] + 2(\varepsilon - \mu B)\mathbf{b}\cdot(\nabla \times \boldsymbol{\kappa})\Big\} \ ,$$

in agreement with the conventional analysis [Hazeltine, Plasma Phys. 15, 77 (1973)].

MOMENTS OF THE DRIFT-KINETIC EQUATION

IN ORDER TO OBTAIN FLUID EQUATIONS FOR THE CGL VARIABLES BY TAKING MOMENTS OF THE DRIFT-KINETIC EQUATION, IT IS MOST ADVANTAGEOUS TO WORK IN THE MACROSCOPIC FLOW REFERENCE FRAME. THEREFORE, SET $\mathbf{u} = n^{-1} \int d^3\mathbf{v}\ \mathbf{v}\ f$ AND $\mathbf{F} = \nabla \cdot \mathsf{P}$

IN THE COEFFICIENT FUNCTIONS OF THE DRIFT-KINETIC EQUATION, ONE NEEDS TO EVALUATE:

$$\frac{\mathbf{b}\times\mathbf{F}}{\Omega_c} = \frac{\mathbf{b}\times(\nabla\cdot\mathsf{P}^{CGL})}{\Omega_c} = \frac{1}{\Omega_c}\left[\mathbf{b}\times\nabla p_\perp + (p_\parallel - p_\perp)\mathbf{b}\times\boldsymbol{\kappa}\right]$$

and

$$\begin{aligned} \mathbf{b}\cdot\mathbf{F} \ &= \ \mathbf{b}\cdot[\nabla\cdot(\mathsf{P}^{CGL}+\mathsf{P}^{gyr})] \ = \ \mathbf{b}\cdot\nabla p_\parallel - (p_\parallel - p_\perp)\mathbf{b}\cdot\nabla\ln B \, + \\ &+ \nabla\cdot\{\frac{1}{\Omega_c}\,\mathbf{b}\times\big[2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + p_\perp\mathbf{b}\times\boldsymbol{\omega} + \nabla(q_\parallel - q_{B\parallel}) + 2(2q_{B\parallel} - q_\parallel)\boldsymbol{\kappa}\big]\} \, + \\ &+ \frac{\mathbf{b}\times\boldsymbol{\kappa}}{\Omega_c}\cdot\big[2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u} + p_\perp\mathbf{b}\times\boldsymbol{\omega} + \nabla(q_\parallel - q_{B\parallel})\big]\} \ + \ p_\perp\sigma \ . \end{aligned}$$

TAKING THE $1,\ v'_\parallel,\ v'^2_\parallel,\ v'^2_\perp,\ v'^3_\parallel,\ v'_\parallel v'^2_\perp$ MOMENTS OF THE FLR DRIFT-KINETIC EQUATION, ONE GETS:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 ,$$

$$\int d^3\mathbf{v}'\, v'_\parallel\, \bar{f} = 0 ,$$

TWO PRESSURE EVOLUTION EQUATIONS IDENTICAL TO THOSE OBTAINED FROM THE MOMENTS OF THE VLASOV EQUATION AND SHOWN EARLIER.

TWO PARALLEL HEAT FLUX EVOLUTION EQUATIONS IDENTICAL TO THOSE OBTAINED FROM THE MOMENTS OF THE VLASOV EQUATION [Ramos, Phys. Plasmas 12, 052102 (2005)].

THE FLUID AND DRIFT-KINETIC EQUATIONS SHOWN MAKE THE FOLLOWING DIAGRAM COMMUTATIVE, INCLUDING THE FIRST-ORDER FLR TERMS FOR SONIC-SCALE TIME EVOLUTION AND MEAN FLOWS:

$$\begin{tikzcd}
\boxed{\text{VLASOV-BOLTZMAN KINETIC EQUATION}} \arrow[r, "\frac{1}{\Omega_c}\ \text{EXPANSION}"] \arrow[d, "\text{MOMENTS}"] & \boxed{\text{DRIFT-KINETIC EQUATION}} \arrow[d, "\text{MOMENTS}"] \\
\boxed{\text{FLUID MOMENT EQUATIONS}} \arrow[r, "\frac{1}{\Omega_c}\ \text{EXPANSION}"] & \boxed{\text{MAGNETIZED FLUID EQUATIONS}}
\end{tikzcd}$$

A FINITE-LARMOR-RADIUS FORM OF THE DRIFT-KINETIC EQUATION HAS BEEN DERIVED, SUITABLE FOR DETERMINATION OF THE FLUID CLOSURES

*** ACCURATE TO THE FIRST FLR ORDER IN THE FAST DYNAMICS ORDERING AND VALID FOR SONIC MACROSCOPIC FLOWS.**

*** USE OF THE FULL MACROSCOPIC FLOW VELOCITY, $\mathbf{u}(\mathbf{x}, t)$, TO DEFINE THE MOVING FRAME. EXACT ALGEBRAIC ELIMINATION OF THE ELECTRIC FIELD AND NO REFERENCE TO $\mathbf{u}_E$ OR ANY OTHER DRIFTS.**

*** FORMULATION IN TERMS OF THE STANDARD MHD VARIABLES (MACROSCOPIC FLOW VELOCITY AND MAGNETIC FIELD).**

*** VELOCITY MOMENTS REPRODUCE ALL THE PREVIOUSLY DERIVED FLUID RESULTS, INCLUDING THE HIGHER-MOMENT FLR RESULTS.**

2007 APS-DPP Meeting, Orlando FL.

FLUID THEORY OF MAGNETIZED PLASMA DYNAMICS AT LOW COLLISIONALITY*

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*Work supported by the U.S. Department of Energy. Special acknowledgment to the CEMM and CSWIM SciDAC's.

OUTLINE

I. INTRODUCTION: SCOPE OF THE WORK.

II. GENERAL FLUID FORMALISM.

III. TWO-FLUID SYSTEM FOR FAST (SONIC) DYNAMICS.

IV. KINETIC DETERMINATION OF THE FLUID CLOSURES.

V. REDUCED TWO-FLUID SYSTEM FOR SLOW (DRIFT) DYNAMICS.

I. INTRODUCTION: SCOPE OF THE WORK

FINITE-LARMOR-RADIUS, TWO-FLUID EQUATIONS TO DESCRIBE MACROSCOPIC DYNAMICS IN MAGNETIZED PLASMAS (EXTENDED MHD):

$$L \sim \rho_\iota/\delta \gg \rho_\iota \ , \qquad \omega_* \sim \delta^2\Omega_{c\iota} \lesssim \partial/\partial t \lesssim \omega_s \sim \delta\Omega_{c\iota}$$

RIGOROUS FLUID CLOSURE ONLY AT HIGH COLLISIONALITY.

IN LOW-COLLISIONALITY REGIMES, THE FLUID MOMENTS OF THE KINETIC EQUATION STILL PROVIDE A GOOD APPROXIMATION FOR THE PERPENDICULAR DYNAMICS.

HYBRID APPROACH: EXPLOIT THE FLUID MOMENT INFORMATION TO THE MAXIMUM, TO BE COMPLEMENTED BY A KINETIC EVALUATION OF THE UNAVAILABLE CLOSURE TERMS.

NOTEWORTHY FEATURES

GENERAL GEOMETRY AND INHOMOGENEOUS MAGNETIC FIELDS.

FULLY NONLINEAR. ARBITRARY FLUCTUATION AMPLITUDES.

FULLY ELECTROMAGNETIC.

FAR-FROM-MAXWELLIAN DISTRIBUTION FUNCTIONS ALLOWED.

EITHER FAST (SONIC) OR SLOW (DIAMAGNETIC DRIFT) ORDERINGS.

DIAMAGNETIC EFFECTS FOR ARBITRARY DENSITY AND TEMPERATURES.

WORKING ASSUMPTIONS IN THE APPLICATIONS TO BE PRESENTED

MACROSCOPIC PHYSICS IN MAGNETIZED PLASMAS: $\delta \sim \rho_\iota / L \ll 1$.

ORDERING OF TIME DERIVATIVES LINKED TO MEAN FLOW VELOCITIES THROUGH $\partial / \partial t \sim u_\alpha / L$, with $\delta v_{th\iota} \lesssim u_\iota \sim u_e \lesssim v_{th\iota}$.

QUASINEUTRALITY WITH SINGLE ION SPECIES: $n_\iota = n_e = n$.

SMALL MASS RATIO: $(m_e/m_\iota)^{1/2} \lesssim \delta$.

LOW COLLISIONALITY: $\nu_\iota / \Omega_{c\iota} \lesssim \delta \, (m_e/m_\iota)^{1/2}$, or $v_{th\iota}/\nu_\iota \sim v_{the}/\nu_e \gtrsim (m_\iota/m_e)^{1/2} L$.

COMPARABLE PRESSURES. ARBITRARY ANISOTROPIES: $p_\iota \sim p_e \sim (p_{\alpha\|} - p_{\alpha\perp})$.

II. GENERAL FLUID FORMALISM

THE RESULTS GIVEN IN THIS SECTION ARE EXACT WITHOUT APPROXIMATIONS AND VALID FOR EACH SPECIES INDEPENDENTLY.

ASSUME ONLY THE UNDERLYING KINETIC DESCRIPTION:

$$\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial t} + v_j \frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial x_j} + \frac{e}{m}(E_j + \epsilon_{jkl} v_k B_l) \frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial v_j} = C(\mathbf{v},\mathbf{x},t),$$

WHERE $C(\mathrm{v}, \mathrm{x}, t)$ IS THE COMPLETE FOKKER-PLANK COLLISION OPERATOR.

DEFINE THE FOLLOWING FLUID MOMENTS:

$$n(\mathbf{x},t) \;=\; \int d^3\mathbf{v}\; f(\mathbf{v},\mathbf{x},t),$$

$$n(\mathbf{x},t)\; u_j(\mathbf{x},t) \;=\; \int d^3\mathbf{v}\; v_j\; f(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{P}_{jk}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)\; f(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{Q}_{jkl}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)(v_l-u_l)\; f(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{R}_{jklm}(\mathbf{x},t) \;=\; m^2\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)(v_l-u_l)(v_m-u_m)\; f(\mathbf{v},\mathbf{x},t) \;=\; \frac{1}{n}\,\mathsf{P}_{[jk}\,\mathsf{P}_{lm]} \;+\; \tilde{\mathsf{R}}_{jklm}(\mathbf{x},t),$$

$$F_j^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)\; C(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{G}_{jk}^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)\; C(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{H}_{jkl}^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)(v_l-u_l)\; C(\mathbf{v},\mathbf{x},t).$$

TAKING THE VELOCITY MOMENTS OF THE KINETIC EQUATION:

$$\frac{\partial n}{\partial t} + \frac{\partial (n u_j)}{\partial x_j} = 0,$$

$$mn\Big(\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k}\Big) + \frac{\partial \mathsf{P}_{jk}}{\partial x_k} - en(E_j + \epsilon_{jkl} u_k B_l) - F_j^{coll} = 0,$$

$$\frac{\partial \mathsf{P}_{jk}}{\partial t} + \frac{\partial}{\partial x_l}\Big(\mathsf{P}_{jk} u_l + \mathsf{Q}_{jkl}\Big) + \frac{\partial u_{[j}}{\partial x_l}\mathsf{P}_{lk]} - \frac{e}{m}\epsilon_{[jlm} B_m \mathsf{P}_{lk]} - \mathsf{G}_{jk}^{coll} = 0,$$

$$\frac{\partial \mathsf{Q}_{jkl}}{\partial t} + \frac{\partial}{\partial x_m}\Big(\mathsf{Q}_{jkl} u_m + \frac{1}{m}\tilde{\mathsf{R}}_{jklm}\Big) + \frac{\partial u_{[j}}{\partial x_m}\mathsf{Q}_{mkl]} - \frac{e}{m}\epsilon_{[jmn} B_n \mathsf{Q}_{mkl]} +$$

$$+ \frac{1}{m}\mathsf{P}_{[jm}\frac{\partial}{\partial x_m}\Big(\frac{1}{n}\mathsf{P}_{kl]}\Big) + \frac{1}{mn} F_{[j}^{coll} \mathsf{P}_{kl]} - \mathsf{H}_{jkl}^{coll} = 0.$$

FORMAL SOLUTION FOR THE STRESS TENSOR

$$\mathsf{P}_{jk} \;=\; p_{\perp}\delta_{jk} \;+\; (p_{\parallel} - p_{\perp}) b_j b_k \;+\; \hat{\mathsf{P}}_{jk} \;=\; \mathsf{P}^{CGL}_{jk} \;+\; \hat{\mathsf{P}}_{jk}$$

with $p \;=\; (p_{\parallel} + 2p_{\perp})/3$ **and** $\hat{\mathsf{P}}_{jj} \;=\; \hat{\mathsf{P}}_{jk} b_j b_k \;=\; 0\,.$

Then,

$$\frac{3}{2}\Big[\frac{\partial p}{\partial t} + \frac{\partial (p u_j)}{\partial x_j}\Big] \;+\; \mathsf{P}_{jk}\frac{\partial u_j}{\partial x_k} \;+\; \frac{\partial q_j}{\partial x_j} \;-\; g^{coll} \;=\; 0\,,$$

$$\frac{1}{2}\Big[\frac{\partial p_{\parallel}}{\partial t} + \frac{\partial (p_{\parallel} u_j)}{\partial x_j}\Big] \;-\; \mathsf{P}_{jk} b_j \Big[\frac{\partial b_k}{\partial t} + u_l \frac{\partial b_k}{\partial x_l} - b_l \frac{\partial u_l}{\partial x_k}\Big] \;+\; \frac{\partial q_{Bj}}{\partial x_j} \;-\; \mathsf{Q}_{jkl} b_j \frac{\partial b_k}{\partial x_l} \;-\; g^{coll}_{B} \;=\; 0\,,$$

and

$$\hat{\mathsf{P}}_{jk} \;=\; \frac{1}{4}\epsilon_{[jlm} b_l \mathsf{K}_{mn}(\delta_{nk]} + 3 b_n b_{k]})$$

with $$\mathsf{K}_{jk} \;=\; \frac{m}{eB}\Big[\frac{\partial \mathsf{P}_{jk}}{\partial t} \;+\; \frac{\partial}{\partial x_l}\big(\mathsf{P}_{jk} u_l + \mathsf{Q}_{jkl}\big) \;+\; \frac{\partial u_{[j}}{\partial x_l}\mathsf{P}_{lk]} \;-\; \mathsf{G}^{coll}_{jk}\Big]\,.$$

A COMPLETELY ANALOGOUS FORMAL SOLUTION IS OBTAINED FOR THE STRESS-FLUX TENSOR:

$$\mathsf{Q}_{jkl} \;=\; q_{T\|}\delta_{[jk}b_{l]} \;+\; (2q_{B\|} - 3q_{T\|})b_jb_kb_l \;+\; \hat{\mathsf{Q}}_{jkl} \;=\; \mathsf{Q}^{CGL}_{jkl} \;+\; \hat{\mathsf{Q}}_{jkl}$$

with $q_{\|} \;=\; q_{B\|} + q_{T\|}$ **and** $\hat{\mathsf{Q}}_{jkk}b_j \;=\; \hat{\mathsf{Q}}_{jkl}b_jb_kb_l \;=\; 0\;.$

$$\frac{\partial q_{\|}}{\partial t} + \frac{\partial(q_{\|}u_j)}{\partial x_j} \;-\; q_j\Big[\frac{\partial b_j}{\partial t} + u_k\frac{\partial b_j}{\partial x_k} - b_k\frac{\partial u_k}{\partial x_j}\Big] \;+\; \mathsf{Q}_{jkl}b_j\frac{\partial u_k}{\partial x_l} \;+\; \frac{1}{m}\mathsf{P}_{jk}b_l\frac{\partial}{\partial x_j}\Big(\frac{3p}{2n}\delta_{kl} + \frac{1}{n}\mathsf{P}_{kl}\Big) \;+$$

$$+\;\frac{1}{2m}b_j\frac{\partial\tilde{\mathsf{R}}_{jkll}}{\partial x_k} \;+\; \frac{1}{mn}b_jF^{coll}_k\Big(\frac{3p}{2}\delta_{jk} + \mathsf{P}_{jk}\Big) \;-\; h^{coll} \;=\; 0\;,$$

etc.

THE TENSOR $\tilde{\mathsf{R}}_{jklm}$ AND THE COLLISIONAL MOMENTS ARE THE KINETIC CLOSURE VARIABLES.

III. TWO-FLUID SYSTEM FOR FAST (SONIC) DYNAMICS

IN A SINGLE-ION QUASINEUTRAL PLASMA:

$$n_e \ = \ n_\imath \ = \ n \ ,$$

$$\frac{\partial n}{\partial t} \ + \ \nabla \cdot (n\mathbf{u}_\imath) \ = \ 0 \ ,$$

$$\mathbf{u}_e \ = \ \mathbf{u}_\imath \ - \ \frac{1}{en}\mathbf{j} \ ,$$

$$\mathbf{j} \ = \ \nabla \times \mathbf{B} \ ,$$

$$\frac{\partial \mathbf{B}}{\partial t} \ + \ \nabla \times \mathbf{E} \ = \ 0 \ .$$

CONSIDER NOW THE FAST ORDERING $\partial/\partial t \sim u_\alpha/L \sim v_{th\imath}/L$, with $L \sim L_\perp \sim L_\parallel$.

ELECTRON MOMENTUM EQUATION, KEEPING $O(v_{th\iota}B) + O(\delta v_{th\iota}B)$:

$$\mathbf{E} \;=\; -\mathbf{u}_\iota \times \mathbf{B} \;+\; \frac{1}{en}\Big(\mathbf{j} \times \mathbf{B} - \nabla \cdot \mathsf{P}_e^{CGL}\Big) \;,$$

with

$$\nabla \cdot \mathsf{P}_\alpha^{CGL} \;=\; \nabla p_{\alpha\perp} \;+\; (\mathbf{B} \cdot \nabla)\Big(\frac{p_{\alpha\parallel} - p_{\alpha\perp}}{B^2}\,\mathbf{B}\Big) \;.$$

THE PARALLEL ELECTRIC FIELD IS AVAILABLE TO $O(\delta v_{th\iota}B) + O(\delta^2 v_{th\iota}B)$:

$$E_\parallel \;=\; \frac{1}{en}\Big[-\mathbf{b} \cdot \nabla p_{e\parallel} \;+\; (p_{e\parallel} - p_{e\perp})\mathbf{b} \cdot \nabla(\ln B) \;+\; F_{e\parallel}^{coll}\Big] \;,$$

with

$$F_{e\parallel}^{coll} \;=\; -\,\frac{\nu_e p_e v_{the}}{n} \int d^3\mathbf{v}\; \frac{\mathbf{b} \cdot (\mathbf{v} - \mathbf{u}_e)}{|\mathbf{v} - \mathbf{u}_e|^3}\; f_e^{(0)}(\mathbf{v}) \;\sim\; \nu_e p_e / v_{the} \;\lesssim\; \delta\; p_e/L \;.$$

SUM OF THE ION AND ELECTRON MOMENTUM EQUATIONS, KEEPING $O(m_\iota n v_{th\iota}^2/L) + O(\delta m_\iota n v_{th\iota}^2/L)$:

$$m_\iota n\Big[\frac{\partial \mathbf{u}_\iota}{\partial t} + (\mathbf{u}_\iota \cdot \nabla)\mathbf{u}_\iota\Big] \;+\; \nabla \cdot \left(\mathsf{P}_e^{CGL} + \mathsf{P}_\iota^{CGL} + \hat{\mathsf{P}}_\iota\right) \;-\; \mathbf{j} \times \mathbf{B} \;=\; 0\;,$$

where

$$\hat{\mathsf{P}}_{\iota,jk} \;=\; \frac{1}{4}\,\epsilon_{[jlm} b_l\;\mathsf{K}_{\iota,mn}\;(\delta_{nk]} + 3b_n b_{k]})$$

and

$$\mathsf{K}_{\iota,mn} \;=\; \frac{m_\iota}{eB}\,\Big\{\,p_{\iota\perp}\frac{\partial u_{\iota,n]}}{\partial x_{[m}} \;+\; \frac{\partial(q_{\iota T\|} b_{n]})}{\partial x_{[m}} \;+\; b_{[m}\,\Big[(2q_{\iota B\|} - 3q_{\iota T\|})\;\kappa_{n]} \;+\; 2(p_{\iota\|} - p_{\iota\perp})\;b_l\frac{\partial u_{\iota,n]}}{\partial x_l}\Big]\,\Big\}\;,$$

with $\boldsymbol{\kappa} = (\mathbf{b} \cdot \nabla)\mathbf{b}$.

FOR THE FIRST TERM OF THE ION GYROVISCOUS STRESS TENSOR,

$$\hat{\mathsf{P}}^{1}_{\iota,jk} \;=\; \frac{m_\iota p_{\iota\perp}}{4eB}\,\epsilon_{[jlm}b_l\left(\frac{\partial u_{\iota,n}}{\partial x_m}+\frac{\partial u_{\iota,m}}{\partial x_n}\right)\left(\delta_{nk]}+3b_nb_{k]}\right)\,,$$

ITS EXACT DIVERGENCE IS

$$\nabla\cdot\hat{\mathsf{P}}^{1}_{\iota} \;=\; -m_\iota n(\mathbf{u}_{*\iota}\cdot\nabla)\mathbf{u}_\iota \;-\; \nabla\chi_\iota \;+\; (\mathbf{B}\cdot\nabla)\Big\{\frac{\chi_\iota}{B^2}\,\mathbf{B} \;+\; \frac{m_\iota p_{\iota\perp}}{eB^2}\,\mathbf{b}\times\left[3(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+\mathbf{b}\times\boldsymbol{\omega}_\iota\right]\Big\} -$$
$$-\;\nabla\times\Big\{\frac{m_\iota p_{\iota\perp}}{2eB}\Big[\,2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota \;+\; \Big(\nabla\cdot\mathbf{u}_\iota-3\mathbf{b}\cdot\left[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\right]\Big)\,\mathbf{b}\,\Big]\Big\}\,,$$

where

$$\mathbf{u}_{*\iota} \;=\; -\frac{1}{en}\nabla\times\left(\frac{p_{\iota\perp}}{B}\mathbf{b}\right)\,, \qquad \chi_\iota \;=\; \frac{m_\iota p_{\iota\perp}}{2eB}(\mathbf{b}\cdot\boldsymbol{\omega}_\iota) \quad \textbf{and} \quad \boldsymbol{\omega}_\iota \;=\; \nabla\times\mathbf{u}_\iota.$$

SIMILAR EXPRESSIONS ARE OBTAINED FOR THE OTHER GYROVISCOSITY TERMS.

ION AND ELECTRON PRESSURE EQUATIONS, KEEPING $O(p_\alpha v_{th\iota}/L)+O(\delta p_\alpha v_{th\iota}/L)$:

$$\frac{3}{2}\Big[\frac{\partial p_\alpha}{\partial t}+\nabla\cdot\big(p_\alpha\mathbf{u}_\alpha\big)\Big] \;+\; p_\alpha\nabla\cdot\mathbf{u}_\alpha \;+\; (p_{\alpha\parallel}-p_{\alpha\perp})\Big\{\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\alpha\big]-\nabla\cdot\mathbf{u}_\alpha/3\Big\} \;+\; \nabla\cdot\big(q_{\alpha\parallel}\mathbf{b}\big) \;+$$

$$+\;\hat{\mathsf{P}}_\alpha : (\nabla\mathbf{u}_\alpha) \;+\; \nabla\cdot\mathbf{q}_{\alpha\perp} \;=\; 0\;,$$

$$\frac{1}{2}\Big[\frac{\partial p_{\alpha\parallel}}{\partial t}+\nabla\cdot\big(p_{\alpha\parallel}\mathbf{u}_\alpha\big)\Big] \;+\; p_{\alpha\parallel}\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\alpha\big] \;+\; \nabla\cdot\big(q_{\alpha B\parallel}\mathbf{b}\big) \;+\; q_{\alpha T\parallel}\mathbf{b}\cdot\nabla(\ln\ B) \;+$$

$$+\;\mathbf{b}\cdot\hat{\mathsf{P}}_\alpha\cdot(\mathbf{b}\times\boldsymbol{\omega}_\alpha) \;+\; \nabla\cdot\mathbf{q}_{\alpha B\perp} \;-\; 2\mathbf{q}_{\alpha B\perp}\cdot\boldsymbol{\kappa} \;+\; q_{\alpha T\parallel}\sigma_\alpha \;-\; g^{coll}_{\alpha B} \;=\; 0\;,$$

where $\hat{\mathsf{P}}_e = 0, \quad \sigma_e = 0,$

$$g^{coll}_{\iota B} \;=\; \frac{\nu_\iota p_\iota v_{th\iota}}{n^2}\int d^3\mathbf{v}\;\frac{|\mathbf{v}-\mathbf{u}_\iota|^2-3[\mathbf{b}\cdot(\mathbf{v}-\mathbf{u}_\iota)]^2}{2|\mathbf{v}-\mathbf{u}_\iota|^3}\;\int d^3\mathbf{w}\;f^{(0)}_\iota(\mathbf{v}+\mathbf{w})\;f^{(0)}_\iota(\mathbf{w})\;\sim\nu_\iota(p_{\iota\parallel}-p_{\iota\perp})$$

and

$$g^{coll}_{eB} = \frac{\nu_e p_e v_{the}}{n}\int d^3\mathbf{v}\;\frac{|\mathbf{v}-\mathbf{u}_e|^2-3[\mathbf{b}\cdot(\mathbf{v}-\mathbf{u}_e)]^2}{2|\mathbf{v}-\mathbf{u}_e|^3}\;\Big[f^{(0)}_e(\mathbf{v})+\frac{1}{n}\int d^3\mathbf{w}\;f^{(0)}_e(\mathbf{v}+\mathbf{w})\;f^{(0)}_e(\mathbf{w})\Big]\;\sim\nu_e(p_{e\parallel}-p_{e\perp})\;.$$

CONTRIBUTIONS OF THE PERPENDICULAR STRESS-FLUX TENSORS TO THE PRESSURE EQUATIONS:

$$\mathbf{q}_{e\perp} \;=\; -\,\frac{1}{eB}\mathbf{b}\times\Big[p_{e\perp}\nabla\Big(\frac{p_{e\parallel}+4p_{e\perp}}{2n}\Big)+\frac{p_{e\parallel}(p_{e\parallel}-p_{e\perp})}{n}\boldsymbol{\kappa}\Big] \;+\; \tilde{\mathbf{q}}_{e\perp}\ ,$$

$$\mathbf{q}_{eB\perp} \;=\; -\,\frac{1}{eB}\mathbf{b}\times\Big[p_{e\perp}\nabla\Big(\frac{p_{e\parallel}}{2n}\Big)+\frac{p_{e\parallel}(p_{e\parallel}-p_{e\perp})}{n}\boldsymbol{\kappa}\Big] \;+\; \tilde{\mathbf{q}}_{eB\perp}\ ,$$

$$\mathbf{q}_{\iota\perp} \;=\; \frac{1}{eB}\mathbf{b}\times\Big[p_{\iota\perp}\nabla\Big(\frac{p_{\iota\parallel}+4p_{\iota\perp}}{2n}\Big)+\frac{p_{\iota\parallel}(p_{\iota\parallel}-p_{\iota\perp})}{n}\boldsymbol{\kappa}+2m_\iota(q_{\iota B\parallel}+2q_{\iota T\parallel})(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+m_\iota q_{\iota T\parallel}\mathbf{b}\times\boldsymbol{\omega}_\iota\Big] \;+\; \tilde{\mathbf{q}}_{\iota\perp}\ ,$$

$$\mathbf{q}_{\iota B\perp} \;=\; \frac{1}{eB}\mathbf{b}\times\Big[p_{\iota\perp}\nabla\Big(\frac{p_{\iota\parallel}}{2n}\Big)+\frac{p_{\iota\parallel}(p_{\iota\parallel}-p_{\iota\perp})}{n}\boldsymbol{\kappa}+2m_\iota q_{\iota B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota+m_\iota q_{\iota T\parallel}\mathbf{b}\times\boldsymbol{\omega}_\iota\Big] \;+\; \tilde{\mathbf{q}}_{\iota B\perp}$$

and

$$q_{\iota T\parallel}\sigma_\iota \;=\; q_{\iota T\parallel}\frac{m_\iota}{4eB}\epsilon_{jkl}b_j\Big(\frac{\partial b_k}{\partial x_m}+\frac{\partial b_m}{\partial x_k}\Big)(\delta_{mn}-b_mb_n)\Big(\frac{\partial u_{\iota,l}}{\partial x_n}+\frac{\partial u_{\iota,n}}{\partial x_l}\Big)\ .$$

IV. KINETIC DETERMINATION OF THE FLUID CLOSURES

1.) THE UNKNOWN TERMS NEEDED TO CLOSE THE TWO-FLUID SYSTEM CAN BE DERIVED FROM MOMENTS OF THE GYROPHASE-AVERAGED $\bar{f}_\alpha$.

2.) THE VELOCITY MOMENTS OF $\bar{f}_\alpha$ NEEDED FOR THE FLUID CLOSURE ARE EVALUATED MOST CONVENIENTLY IN THE MOVING FRAME OF THE FULL MACROSCOPIC FLOW VELOCITY, $\mathbf{u}_\alpha(\mathbf{x}, t)$.

3.) FOR THE FAST DYNAMICS UNDER CONSIDERATION, WITH LARGE PERPENDICULAR ELECTRIC FIELDS $E_\perp \sim v_{th\iota}B$, THE DRIFT-KINETIC EQUATION MUST BE DERIVED IN A MOVING FRAME CLOSE TO THE ELECTRIC DRIFT VELOCITY $\mathbf{u}_E(\mathbf{x}, t) = \mathbf{E} \times \mathbf{B}/B^2$, SUCH AS $\mathbf{u}_E(\mathbf{x}, t)$ ITSELF OR $\mathbf{u}_\alpha(\mathbf{x}, t)$.

4.) TO DETERMINE THE COLLISIONAL MOMENTS AND THE PERPENDICULAR HEAT FLUX CLOSURE TERMS $\tilde{\mathrm{q}}_{\alpha\perp}$ AND $\tilde{\mathrm{q}}_{\alpha B\perp}$ WITHIN THE REQUIRED ACCURACY, ONLY THE LOWEST-ORDER OR ZERO-LARMOR-RADIUS DISTRIBUTION FUNCTIONS $\bar{f}_\alpha = \bar{f}_\alpha^{(0)}$ ARE NEEDED.

5.) TO DETERMINE THE PARALLEL HEAT FLUXES (OR THE COEFFICIENT FUNCTIONS IN THEIR EVOLUTION EQUATIONS) WITHIN THE REQUIRED ACCURACY, FIRST-ORDER FLR SOLUTIONS OF THE DRIFT-KINETIC EQUATION, $\bar{f}_\alpha = \bar{f}_\alpha^{(0)} + \bar{f}_\alpha^{(1)}$, ARE NECESSARY.

A FINITE-LARMOR-RADIUS FORM OF THE DRIFT-KINETIC EQUATION HAS BEEN DERIVED, SUITABLE FOR DETERMINATION OF THE FLUID CLOSURES

*** ACCURATE TO THE FIRST FLR ORDER IN THE FAST DYNAMICS ORDERING AND VALID FOR SONIC MACROSCOPIC FLOWS.**

*** USE OF THE FULL MACROSCOPIC FLOW VELOCITY, $\mathbf{u}_\alpha(\mathbf{x}, t)$, TO DEFINE THE MOVING FRAME. EXACT ALGEBRAIC ELIMINATION OF THE ELECTRIC FIELD AND NO REFERENCE TO $\mathbf{u}_E$ OR ANY OTHER DRIFTS.**

*** FORMULATION IN TERMS OF THE STANDARD MHD VARIABLES (MACROSCOPIC FLOW VELOCITY AND MAGNETIC FIELD).**

*** VELOCITY MOMENTS REPRODUCE ALL THE PREVIOUSLY DERIVED FLUID RESULTS, INCLUDING THE HIGHER-MOMENT FLR RESULTS.**

FINITE-LARMOR-RADIUS, ”DRIFTLESS” DRIFT-KINETIC EQUATION

$$\frac{\partial \bar{f}(v_\parallel', v_\perp', \mathbf{x}, t)}{\partial t} \;+\; \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} \;+\; \dot{v}_\parallel' \, \frac{\partial \bar{f}}{\partial v_\parallel'} \;+\; \dot{v}_\perp' \, \frac{\partial \bar{f}}{\partial v_\perp'} \;=\; \frac{D^{coll} \bar{f}}{Dt} \;,$$

with the coefficient functions:

$$\dot{\mathbf{x}} \;=\; \mathbf{u} + v_\parallel' \mathbf{b} \;+\; \frac{v_\perp'^2}{2} \nabla \times \frac{\mathbf{b}}{\Omega_c} \;-\; \frac{\mathbf{b}}{\Omega_c} \times \Big[\frac{\nabla \cdot \mathsf{P}}{mn} - 2 v_\parallel' (\mathbf{b} \cdot \nabla) \mathbf{u} - \Big(v_\parallel'^2 - \frac{v_\perp'^2}{2} \Big) \boldsymbol{\kappa} \Big] \;,$$

$$\dot{v}_\parallel' \;=\; \frac{\mathbf{b} \cdot (\nabla \cdot \mathsf{P})}{mn} \;-\; v_\parallel' \mathbf{b} \cdot [(\mathbf{b} \cdot \nabla) \mathbf{u}] \;-\; \frac{v_\perp'^2}{2} \mathbf{b} \cdot \nabla \ln B \;+\; \frac{v_\perp'^2}{2} \nabla \cdot \Big[\frac{\mathbf{b}}{\Omega_c} \times (\boldsymbol{\omega} \times \mathbf{b} + v_\parallel' \boldsymbol{\kappa}) \Big] \;+$$
$$+ \Big[\frac{\mathbf{b}}{\Omega_c} \times (\boldsymbol{\omega} \times \mathbf{b} + v_\parallel' \boldsymbol{\kappa}) \Big] \cdot \Big[\frac{\nabla \cdot \mathsf{P}}{mn} - 2 v_\parallel' (\mathbf{b} \cdot \nabla) \mathbf{u} - v_\parallel'^2 \boldsymbol{\kappa} \Big] \;-\; \frac{v_\perp'^2}{2} \sigma \;,$$

$$\dot{v}_\perp' \;=\; \frac{v_\perp'}{2} \Big\{ \mathbf{b} \cdot [(\mathbf{b} \cdot \nabla) \mathbf{u}] \;-\; \nabla \cdot \mathbf{u} \;+\; v_\parallel' \mathbf{b} \cdot \nabla \ln B \;+\; \nabla \cdot \Big[\frac{\mathbf{b}}{\Omega_c} \times \Big(\frac{\nabla \cdot \mathsf{P}}{mn} - 2 v_\parallel' (\mathbf{b} \cdot \nabla) \mathbf{u} - v_\parallel'^2 \boldsymbol{\kappa} \Big) \Big] \;+$$
$$+\, 2 \Big[\frac{\mathbf{b}}{\Omega_c} \times (\boldsymbol{\omega} \times \mathbf{b}) \Big] \cdot \Big[(\mathbf{b} \cdot \nabla) \mathbf{u} + v_\parallel' \boldsymbol{\kappa} \Big] \;-\; \Big(\frac{\mathbf{b}}{\Omega_c} \times \boldsymbol{\kappa} \Big) \cdot \Big[\frac{\nabla \cdot \mathsf{P}}{mn} - 4 v_\parallel' (\mathbf{b} \cdot \nabla) \mathbf{u} \Big] \Big\} \;.$$

THE FLUID AND DRIFT-KINETIC EQUATIONS SHOWN MAKE THE FOLLOWING DIAGRAM COMMUTATIVE, INCLUDING THE FIRST-ORDER FLR TERMS FOR SONIC-SCALE TIME EVOLUTION AND MEAN FLOWS:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{VLASOV-BOLTZMAN}\\ \text{KINETIC EQUATION}\end{array}} \arrow[r, "\frac{1}{\Omega_c}\ \text{EXPANSION}"] \arrow[d, "\text{MOMENTS}"] & \boxed{\begin{array}{c}\text{DRIFT-KINETIC}\\ \text{EQUATION}\end{array}} \arrow[d, "\text{MOMENTS}"] \\
\boxed{\begin{array}{c}\text{FLUID MOMENT}\\ \text{EQUATIONS}\end{array}} \arrow[r, "\frac{1}{\Omega_c}\ \text{EXPANSION}"] & \boxed{\begin{array}{c}\text{MAGNETIZED}\\ \text{FLUID EQUATIONS}\end{array}}
\end{tikzcd}$$

V. REDUCED TWO-FLUID SYSTEM FOR SLOW (DRIFT) DYNAMICS

A WIDELY USED ORDERING FOR SLOW DYNAMICS ON THE DIAMAGNETIC DRIFT SCALE IS:

$$\partial/\partial t \ \sim \ \omega_{*\alpha} \ \sim \ \delta^2 \, \Omega_{ci} \ ,$$

$$u_\alpha \ \sim \ u_{*\alpha} \ \sim \ \delta \, v_{thi} \ .$$

CONSISTENT IMPLEMENTATION OF THIS DRIFT ORDERING REQUIRES EITHER A SECOND-ORDER-ACCURATE EVALUATION OF THE CGL PRESSURES [i.e. $p_{\iota\|,\perp} = O(m_\iota n v_{th\iota}^2) + O(\delta^2 m_\iota n v_{th\iota}^2)$ WHICH CAN ONLY BE ACCOMPLISHED KINETICALLY], OR THE INTRODUCTION OF A SMALL PARALLEL GRADIENT [$\mathbf{b} \cdot \nabla = O(\delta^2/L)$] SUBSIDIARY ORDERING.

DRIFT-ORDERED TWO-FLUID REDUCED SYSTEM WITH $\nabla_\parallel \sim \delta^2/L$

Reduced fluid systems are characterized by:

Separate parallel and perpendicular length scales in large aspect ratio geometry.

Subsidiary expansion parameter $\epsilon \sim L_\perp/L_\parallel \sim k_\parallel/k_\perp \ll 1$ besides $\delta \sim \rho_\iota/L_\perp \ll 1$.

Weakly inhomogeneous magnetic field ($||\nabla \mathrm{B}|| \sim B/L_\parallel$).

Elimination of the fast magnetosonic wave.

Carry out one such reduction of our low-collisionality two-fluid equations, assumming:

Slow dynamics, with drift-ordered particle and heat flows:

$$\partial/\partial t \sim \delta^2\,\Omega_{c\iota}\ , \qquad u_{\alpha\parallel} \sim u_{\alpha\perp} \sim \delta\ v_{th\iota}\ , \qquad q_{\alpha\parallel} \sim q_{\alpha\perp} \sim \delta\ v_{th\iota}\ p_\alpha\ .$$

Weakly inhomogenous magnetic field and small parallel gradient ordering in a large aspect ratio toroidal background geometry, with $\beta \sim \epsilon \sim \delta^2$:

$$\epsilon \sim (R-R_0)/R_0 \sim L_\perp/L_\parallel \sim \delta^2\ , \qquad k_\parallel \sim \mathbf{b}\cdot\nabla \sim \mathbf{e}_\zeta\cdot\nabla \sim 1/R_0 \sim \epsilon k_\perp\ ,$$

$$\mathbf{B}\ =\ B_0\mathbf{e}_\zeta + \mathbf{B}_1\ , \qquad |\mathbf{B}_1| \sim \epsilon B_0\ , \qquad p_\iota \sim p_e\ \sim\ (p_{\alpha\parallel} - p_{\alpha\perp})\ \sim\ \epsilon B_0^2\ .$$

TWO-FLUID REDUCED SYSTEM FEATURES

Valid for general density and temperature gradients and for arbitrary density and temperature fluctuation amplitudes. Rigorous account of all the relevant diamagnetic effects, in particular ion gyroviscosity and ion and electron perpendicular heat fluxes.

All dynamical fields advected by the leading-order $\mathbf{u}_E$ drift: $d'/dt = \partial/\partial t + B_0^{-1}[\Phi,\ ]$.

Parallel heat flux terms absent from the lowest-significant-order system, which is formally closed except for the three collisional moments $F_{e\|}^{coll}$, g_{eB}^{coll} **and** $g_{\iota B}^{coll}$.

Ion parallel velocity evolved by $m_\iota n\ d'u_{\iota\|}/dt + \nabla_\|(p_{\iota\|} + p_{e\|}) = 0$, **decoupled from the rest.**

Use of the parallel curl of the ion particle flux, $W = \mathbf{B}\cdot[\nabla\times(n\mathbf{u}_\iota)]$, **as auxiliary variable to determine the electric potential from the vorticity equation.**

FINAL FORM OF THE TWO-FLUID REDUCED SYSTEM

The coupled system for the seven scalar fields n , $p_{\alpha\parallel}$, $p_{\alpha\perp}$, ψ and Φ , derived from the continuity equation, the four pressure evolution equations and the parallel components of the generalized Ohm's law and the curl of the total momentum equation in their lowest significant order, is:

$$\frac{d'n}{dt} \;=\; 0 \,,$$

$$\frac{1}{2}\frac{d'p_{\alpha\parallel}}{dt} \;-\; g_{\alpha B}^{coll} \;=\; 0 \,,$$

$$\frac{d'p_{\alpha\perp}}{dt} \;+\; \left(\frac{p_{\alpha\parallel}-p_{\alpha\perp}}{3e_{\alpha}B_0n^2}\right)\,[n,p_{\alpha\perp}] \;+\; g_{\alpha B}^{coll} \;=\; 0 \,,$$

$$\frac{\partial\psi}{\partial t} \;+\; \nabla_{\parallel}\Phi \;-\; \frac{1}{en}\nabla_{\parallel}\,p_{e\parallel} \;+\; \frac{1}{en}\,F_{e\parallel}^{coll} \;=\; 0 \,,$$

$$\nabla_{\perp}\cdot(n\nabla_{\perp}\Phi) \;=\; W \;-\; \frac{1}{e}\,\nabla_{\perp}^2 p_{\iota\perp} \,,$$

$$\frac{d'W}{dt} \;+\; \frac{1}{2B_0}[|\nabla_{\perp}\Phi|^2,n] \;+\; \frac{B_0^2}{m_{\iota}}\nabla_{\parallel}(\nabla_{\perp}^2\psi) \;+\; \frac{1}{eB_0}[\nabla_{\perp}\Phi;\nabla_{\perp}p_{\iota\perp}] \;+\; \frac{B_0}{m_{\iota}R_0}[R,(p_{\iota\parallel}+p_{\iota\perp}+p_{e\parallel}+p_{e\perp})] \;=\; 0.$$

HIGHLIGHTED FINITE-LARMOR-RADIUS RESULTS

LOW-COLLISIONALITY GYROVISCOSITY TENSOR FOR GENERAL FLOWS AND ANISOTROPIC PRESSURES. EXACT EXPRESSION OF ITS DIVERGENCE.

PERPENDICULAR HEAT FLUXES FOR GENERAL FLOWS AND ANISOTROPIC PRESSURES.

DYNAMIC EVOLUTION EQUATIONS FOR PARALLEL HEAT FLUXES.

DRIFT-KINETIC EQUATION IN THE MEAN FLOW REFERENCE FRAME.

REDUCED TWO-FLUID SYSTEM IN THE DIAMAGNETIC DRIFT DYNAMICAL SCALE.