Equilibrium Codes and Solvers

Physics status, math, and algorithmic issues

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PPPL

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Outline

I. Solvers
   - Progress on TSC (FSP) block tridiagonal solver with GLF23
   - GCNM

II. Equilibrium Equation with Adiabatic Invariants
   - TEQ Grad/Shafranov q-solver
   - TSC relaxation solver
   - Extensions

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The FSP Solver-1

Model problem to illustrate the need for a Newton Solver with GLF23:

Consider a diffusion equation for the temperature T in toroidal flux coordinates Φ :

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial \Phi}\left[\Phi\ \chi\left(T'\right)\frac{\partial T}{\partial \Phi}\right] + S \qquad (1)$$

We apply a boundary condition of T=0 at Φ=1, keep S=1, and use the above solution as an initial condition. Now, define a function that mimics the critical gradient thermal diffusivity model GLF23:

$$\chi\left(T'\right) = \begin{Bmatrix} k\left(\left|T'\right| - T_c'\right)^{\alpha} + \chi_0 & \text{for}\left|T'\right| > T_c' \\ \chi_0 & \text{for}\left|T'\right| \le T_c' \end{Bmatrix} \qquad (2)$$

For definiteness, let χ_0=*1.0,* α=*0.5, k = 10,* and with $T_c' = 0.5$

The FSP Solver-2

Formulation for solution of model problem with Crank Nicholson:

Try solving this with the θ-implicit method (Crank-Nicholson corresponds to θ=0.5): , with s= $\Delta t / \Delta\Phi^2$

$$T_j^{n+1} = T_j^n + s\theta\left\{\left[\Phi_{j+1/2}\chi\left(T'^n_{j+1/2}\right)\left(T_{j+1}^{n+1} - T_j^{n+1}\right)\right] - \left[\Phi_{j-1/2}\chi\left(T'^n_{j-1/2}\right)\left(T_j^{n+1} - T_{j-1}^{n+1}\right)\right]\right\}$$

$$+ s(1-\theta)\left\{\left[\Phi_{j+1/2}\chi\left(T'^n_{j+1/2}\right)\left(T_{j+1}^n - T_j^n\right)\right] - \left[\Phi_{j-1/2}\chi\left(T'^n_{j-1/2}\right)\left(T_j^n - T_{j-1}^n\right)\right]\right\} + \Delta t S$$

Or,

$$A_j T_{j+1}^{n+1} - B_j T_j^{n+1} + C_j T_{j-1}^{n+1} + D_j = 0$$

$$A_j = s\theta\Phi_{j+1/2}\chi\left(T'^n_{j+1/2}\right)$$

$$C_j = s\theta\Phi_{j-1/2}\chi\left(T'^n_{j-1/2}\right)$$

$$B_j = 1 + A_j + C_j$$

$$D_j = T_j^n + s(1-\theta)\left\{\left[\Phi_{j+1/2}\chi\left(T'^n_{j+1/2}\right)\left(T_{j+1}^n - T_j^n\right)\right] - \left[\Phi_{j-1/2}\chi\left(T'^n_{j-1/2}\right)\left(T_j^n - T_{j-1}^n\right)\right]\right\} + \Delta t S$$

The FSP Solver-3

Results for Solution of model problem with Crank Nicholson:

Fig. 1a: T vs time at zone 10

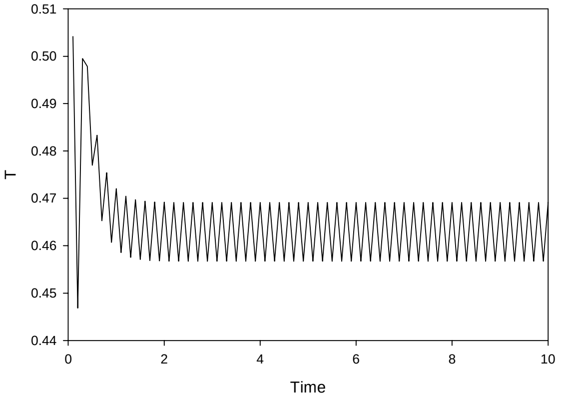
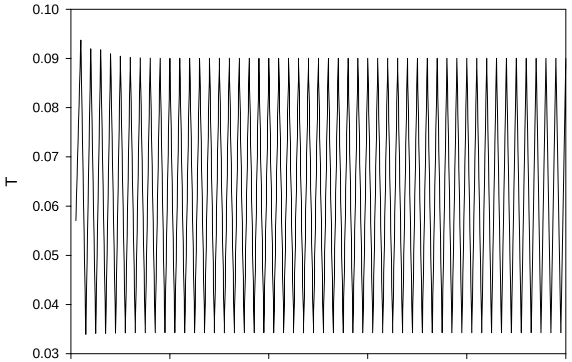


Fig. 1b: T vs time at zone 90



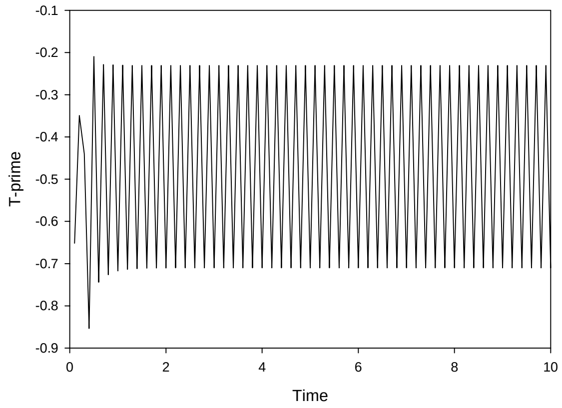
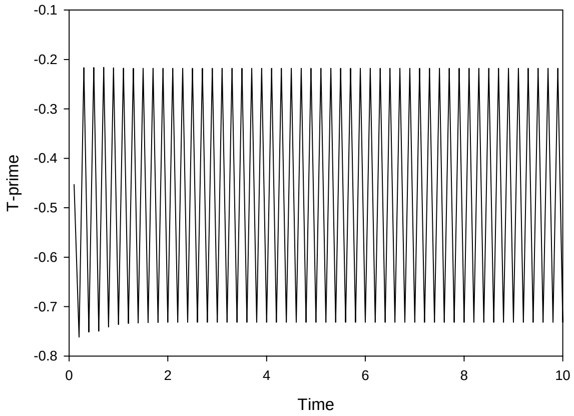


Fig. 2b: T-prime vs time at zone 90



The illustrations in Figs 1 and 2 are the result of solving this with n=100 zones, with a time step Δt=0.1, implicit parameter θ=1.0 for 100 time steps. We plot the time history of the function and derivative at locations 10 and 90.

The FSP Solver-4

Formulation for solution of model problem with nonlinear implicit method:

The nonlinear θ-implicit method corresponds to: (with s=$\Delta t / \Delta\Phi^2$)

$$T_j^{n+1} = T_j^n + s\theta\left\{\left[\Phi_{j+1/2}\chi\left(T'^{n+1}_{j+1/2}\right)\left(T_{j+1}^{n+1} - T_j^{n+1}\right)\right] - \left[\Phi_{j-1/2}\chi\left(T'^{n+1}_{j-1/2}\right)\left(T_j^{n+1} - T_{j-1}^{n+1}\right)\right]\right\}$$
$$+ s\left(1-\theta\right)\left\{\left[\Phi_{j+1/2}\chi\left(T'^{n}_{j+1/2}\right)\left(T_{j+1}^{n} - T_j^{n}\right)\right] - \left[\Phi_{j-1/2}\chi\left(T'^{n}_{j-1/2}\right)\left(T_j^{n} - T_{j-1}^{n}\right)\right]\right\} + \Delta t S$$

Or,

$$A_j T_{j+1}^{n+1} - B_j T_j^{n+1} + C_j T_{j-1}^{n+1} + D_j = 0$$

$$A_j = s\theta\Phi_{j+1/2}\chi\left(T'^{n+1}_{j+1/2}\right)$$
$$C_j = s\theta\Phi_{j-1/2}\chi\left(T'^{n+1}_{j-1/2}\right)$$
$$B_j = 1 + A_j + C_j$$
$$D_j = T_j^n + s\left(1-\theta\right)\left\{\left[\Phi_{j+1/2}\chi\left(T'^{n}_{j+1/2}\right)\left(T_{j+1}^{n} - T_j^{n}\right)\right] - \left[\Phi_{j-1/2}\chi\left(T'^{n}_{j-1/2}\right)\left(T_j^{n} - T_{j-1}^{n}\right)\right]\right\} + \Delta t S$$

The FSP Solver-5

A Newton method for solving the nonlinear implicit equations is still a tri-diagonal system, but involves derivatives of χ wrt *T* and *T'*:

$$A_j T_{j+1}^{n+i/N} - B_j T_j^{n+i/N} + C_j T_{j-1}^{n+i/N} + D_j = 0$$

$$A_j = s\theta\Phi_{j+1/2}\left[\chi_{j+1/2} + \left(\frac{\partial\chi}{\partial T'}\right) T'^{\,n+(i-1)/N}_{j+1/2}\right]$$

$$C_j = s\theta\Phi_{j-1/2}\left[\theta\chi_{j-1/2} + \left(\frac{\partial\chi}{\partial T'}\right) T'^{\,n+(i-1)/N}_{j-1/2}\right]$$

$$B_j = 1 + A_j + C_j$$

$$D_j = T_j^n + s(1-\theta)\left\{\left[\Phi_{j+1/2}\chi_{j+1/2}\left(T_{j+1}^n - T_j^n\right)\right] - \left[\Phi_{j-1/2}\chi_{j-1/2}\left(T_j^n - T_{j-1}^n\right)\right]\right\}$$

$$+ \Delta t S + s\Phi_{j+1/2} T'^{\,n+(i-1)/N}_{j+1/2}\left(T_j^{n+(i-1)/N} - T_{j+1}^{n+(i-1)/N}\right)\left[\frac{\partial\chi}{\partial T'}\right]$$

$$+ s\Phi_{j-1/2} T'^{\,n+(i-1)/N}_{j-1/2}\left(T_j^{n+(i-1)/N} - T_{j-1}^{n+(i-1)/N}\right)\left[\frac{\partial\chi}{\partial T'}\right]$$

The PTRANSP FSP Solver-6

Results for Solution of model problem with Newton Method:

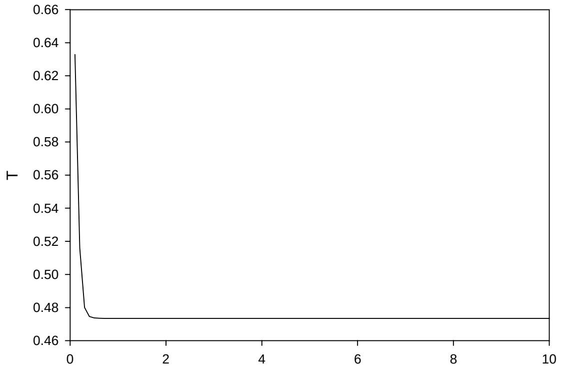


Fig. 3a: T vs time at zone 10

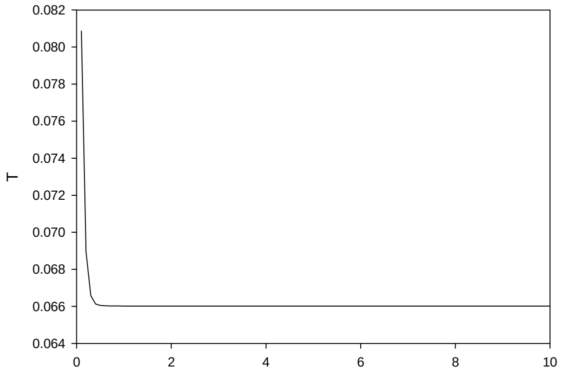


Fig. 3b: T vs time at zone 90

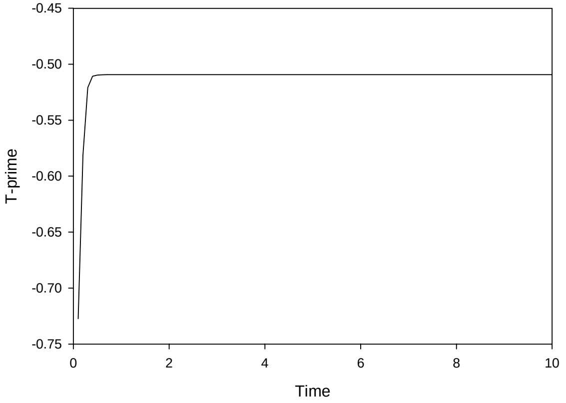


Fig 4a: T-prime vs time at zone 10

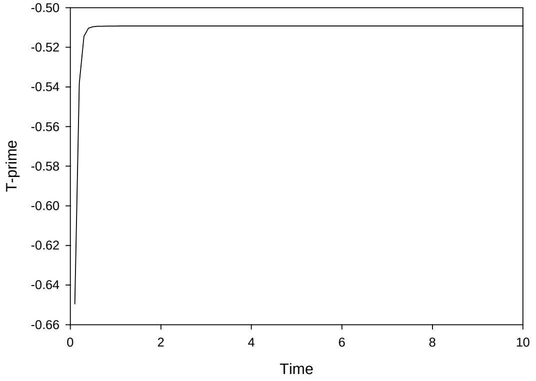


Fig. 4b: T-prime vs time at zone 90

We see that the solution converges to a mean result (compared to Figs 1 and 2) and without oscillations.

The FSP Solver-7

This has been generalized in terms of general geometry and separate ion and electron temperatures, but keeping the same block tri-diagonal structure:

$$\frac{\partial}{\partial t}\sigma = \tfrac{2}{3}\left(V'\right)^{2/3}\left[V_L\frac{\partial K}{\partial \Phi} - \frac{\partial}{\partial \Phi}V'\left(q^i + q^e\right) + V'\left(S_e + S_i - R_e\right)\right]$$

$$\frac{\partial}{\partial t}\sigma_e = \tfrac{2}{3}\left(V'\right)^{2/3}\left[V_L\frac{\partial K}{\partial \Phi} - \frac{\partial}{\partial \Phi}V'\left(q^e\right) + V'\left(-\Gamma\frac{\partial p_i}{\partial \Phi} + Q_{\Delta e} + S_e - R_e\right)\right]$$

NOTE: in previous notation: $Q^i = V'q^i, Q^e = V'q^e$

$$\text{dsi} = q^i = q^{i0} + q^{i1}\frac{\partial}{\partial \Phi}\left(\frac{N}{V'}\right) + q^{i2}\frac{\partial}{\partial \Phi}\left(\frac{\sigma}{V'^{5/3}}\right) + q^{i3}\frac{\partial}{\partial \Phi}\left(\frac{\sigma_e}{V'^{5/3}}\right) + q^{i4}\frac{\partial}{\partial \Phi}\left(\iota\left\langle\frac{J\left|\nabla\psi\right|^2}{R^2}\right\rangle\left\langle\frac{J}{R^2}\right\rangle\right)$$

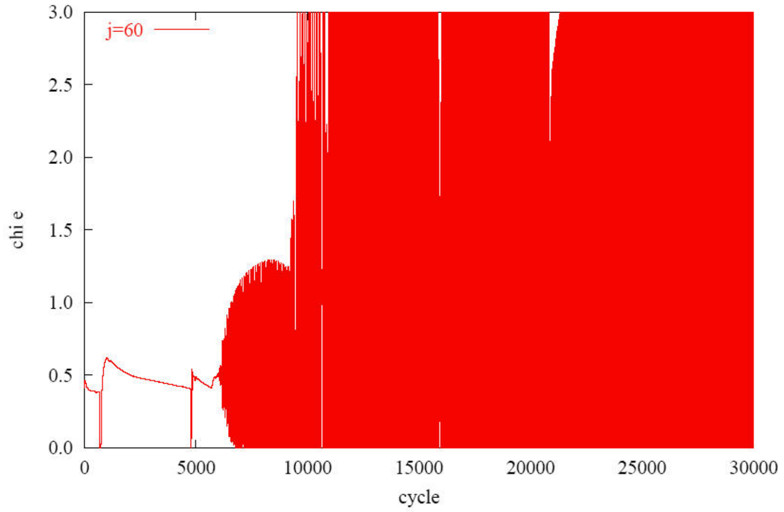
$$\text{dse} = q^e = q^{e0} + q^{e1}\frac{\partial}{\partial \Phi}\left(\frac{N}{V'}\right) + q^{e2}\frac{\partial}{\partial \Phi}\left(\frac{\sigma}{V'^{5/3}}\right) + q^{e3}\frac{\partial}{\partial \Phi}\left(\frac{\sigma_e}{V'^{5/3}}\right) + q^{e4}\frac{\partial}{\partial \Phi}\left(\iota\left\langle\frac{J\left|\nabla\psi\right|^2}{R^2}\right\rangle\left\langle\frac{J}{R^2}\right\rangle\right)$$

$$\sigma = (p_e + p_i)V'^{5/3}$$

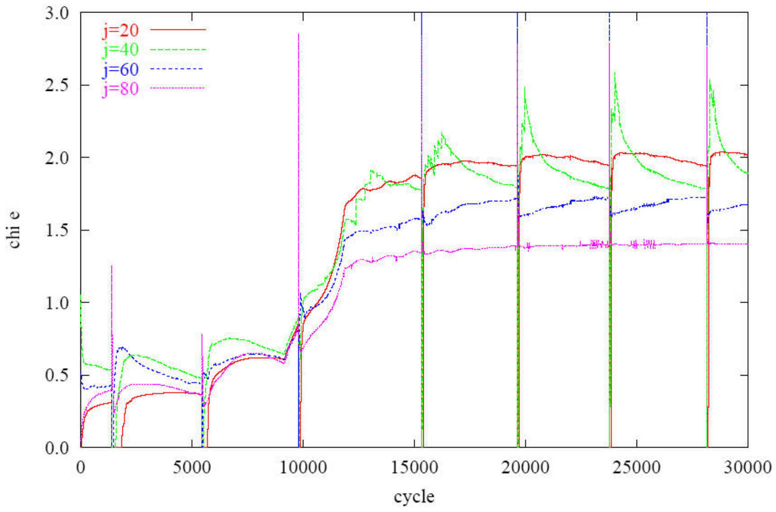
$$\sigma_e = (p_e + p_i)V'^{5/3}$$

The FSP Solver-8

This has been implemented in the full solver:



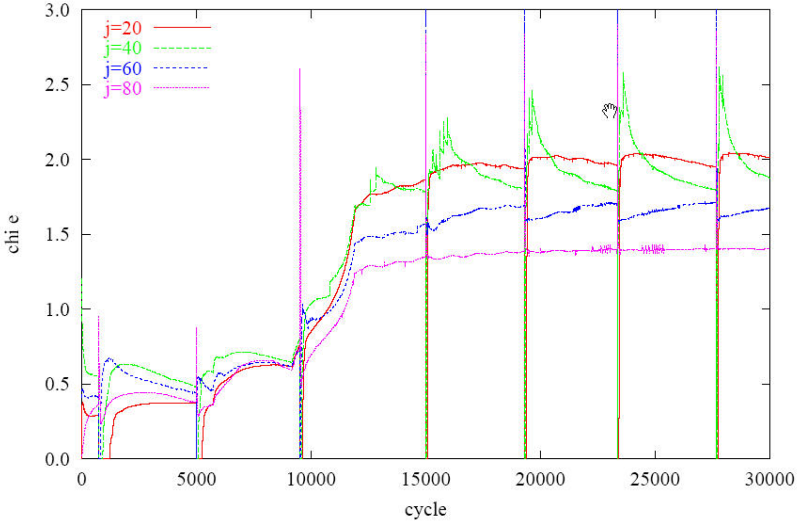
Without linearization



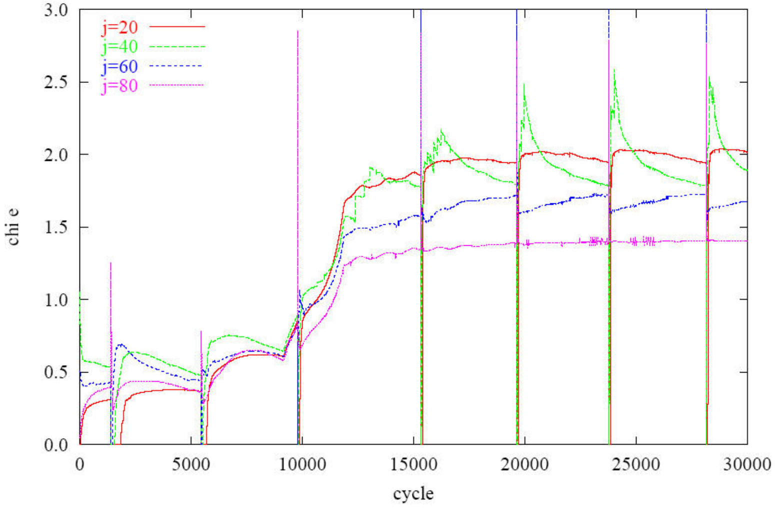
With linearization

The FSP Solver-9

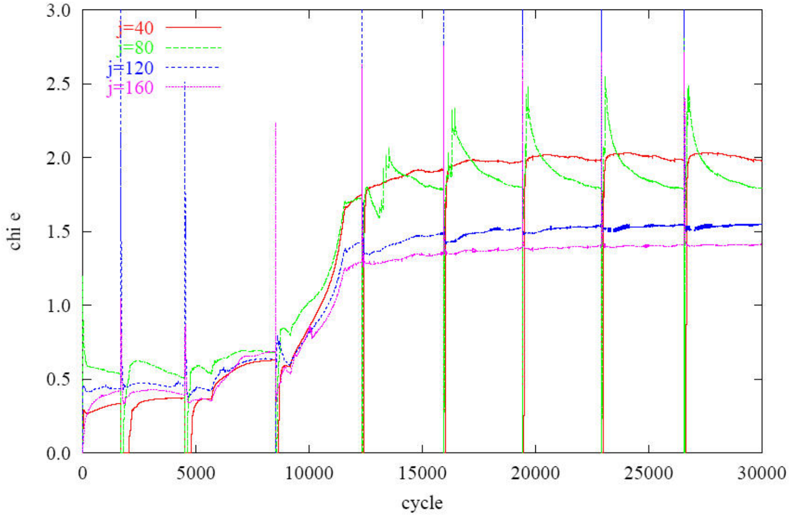
Convergence Tests:



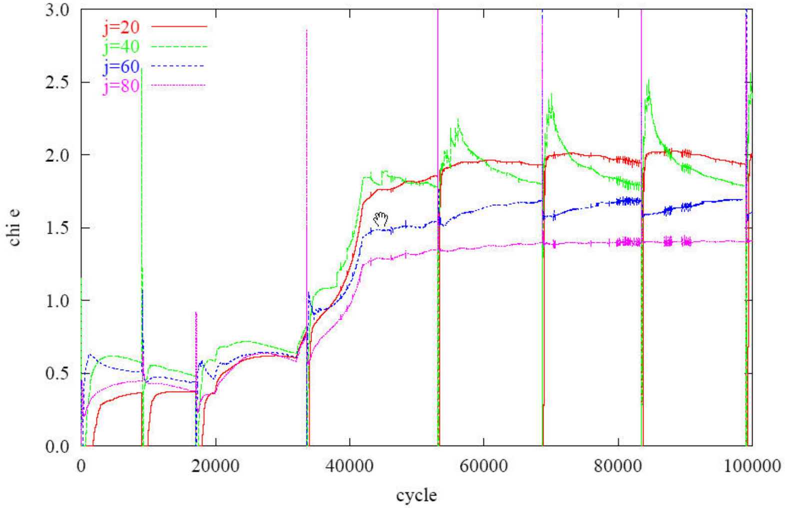
Base case: 1 Newton iteration per timestep



3 Newton iterations per timestep

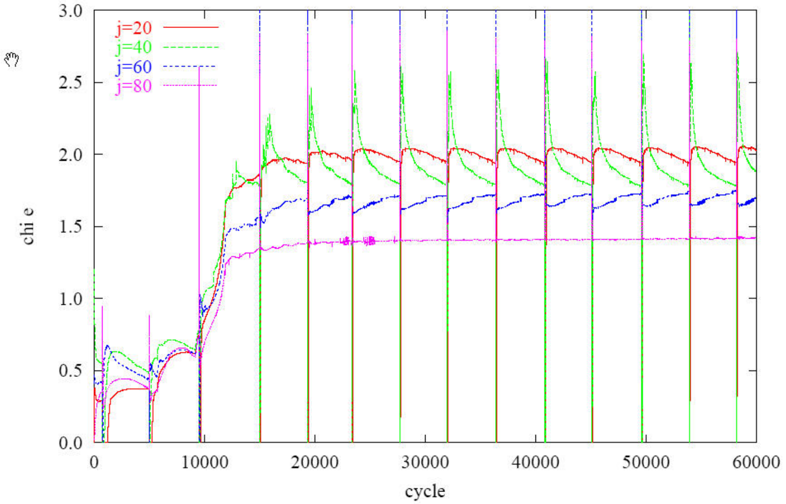


Double # of zones

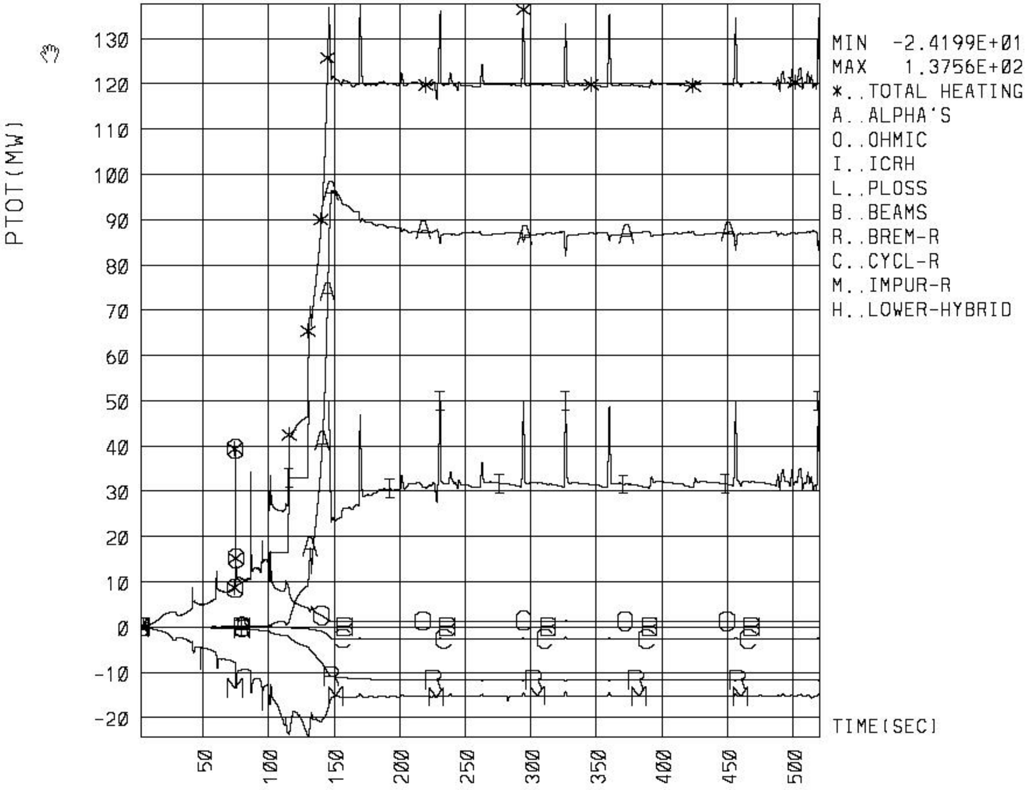


Reduce timestep by 3

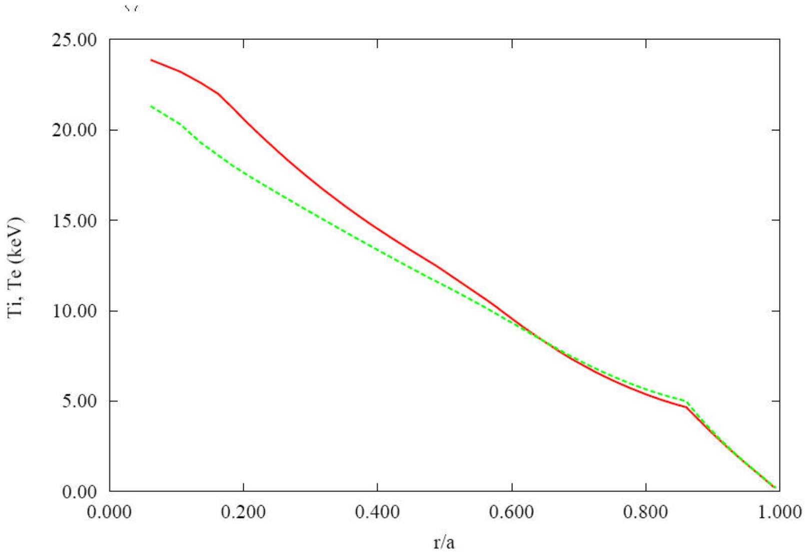
Results for a 500s ITER run:



Chi Values for entire run



Energies vs time



Profiles at 250s

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Globally Convergent Newton’s Method (GCNM)

- The GCNM also applies Newton’s Method to solve the nonlinear time advance equations, but does not retain the tri-diagonal structure when computing the Jacobian matrix
- It is “more robust” than the FSP solver in that it will always return a “best” solution; however it can be considerably more time-consuming for large number of zones
- It is being made parallel
- Has not yet been officially delivered, although a prototype version was offered.

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Equilibrium Equation

In the Evolving Equilibrium description, the momentum equation is replaced by force balance equation:

$$\rho\vec{V}\bullet\nabla\vec{V}+\nabla\bullet\overleftrightarrow{P}=\vec{J}\times\vec{B}$$

For an axisymmetric plasma with $\vec{V}=0$ and $\overleftrightarrow{P}=\mathrm{p}\overleftrightarrow{I}$
this is equivalent to the Grad-Shafranov-Schlüter (GSS) equation:

$$R^2\nabla\bullet\frac{1}{R^2}\nabla\psi+R^2\mu_0 p(\psi)+\frac{1}{2}\frac{d}{d\psi}g^2(\psi)=0$$

Here, we use the notation: $\vec{B}=\nabla\phi\times\nabla\psi+g\nabla\phi$ $\qquad \mu_0\vec{J}=\nabla\times\vec{B}$

$\mu_0 = 4\pi\times10^{-7}$

Note the existence of the 2 “free functions” that must be specified to solve the GSS equation. In this case they are $p(\psi)$ and $g(\psi)$, but these are not always the most physically meaningful. Also, B.C. need to be supplied.

Equilibrium Eq. with Adiabatic Invariants

- The adiabatic invariants should be exactly the same before and after the equilibrium solution is called.
- The way to insure this is to write the equilibrium equation in terms of the adiabatic invariants σ(Φ) and ι(Φ):

$$\text{noting that}: \; p(\Phi) = \frac{\sigma(\Phi)}{V'^{5/3}}, \quad g(\Phi) = \frac{(2\pi)^{-1}}{i(\Phi)V'\left\langle R^{-2}\right\rangle}, \; \text{we have:}$$

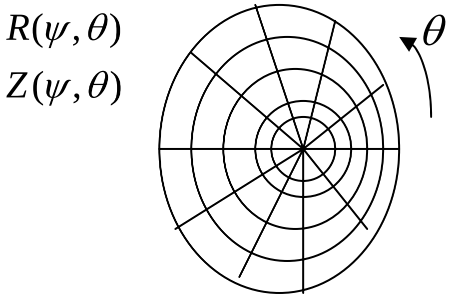
$$R^2\nabla\bullet R^{-2}\nabla\Psi + R^2\frac{d}{d\Psi}\left[\frac{\sigma(\Phi)}{V'^{5/3}}\right] + \frac{(2\pi)^{-2}}{i(\Phi)V'\left\langle R^{-2}\right\rangle}\frac{d}{d\Psi}\left[\frac{1}{\iota(\Phi)V'\left\langle R^{-2}\right\rangle}\right] = 0$$

Note that this integral-differential equation implies that contouring and surface averaging are required during the GSS iterations: QDE.

2-Approaches for Equilibrium

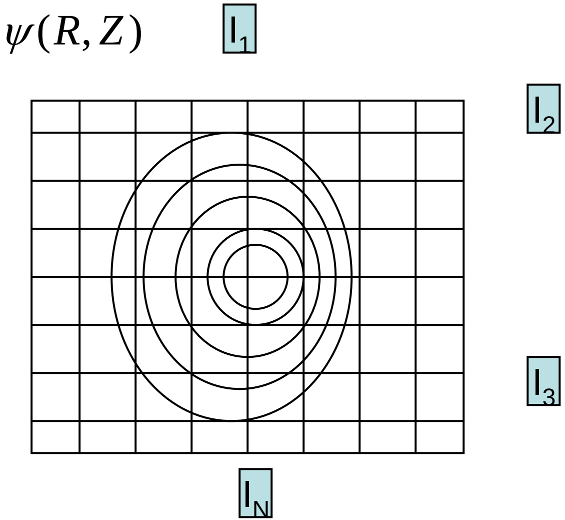
Fixed Boundary (POLAR2D/TEQ)

- Inverse Equilibrium equation solved inside a given, fixed boundary



Free Boundary (TEQ)*

- Equilibrium equation solved everywhere, requires coil currents, determines boundary shape.



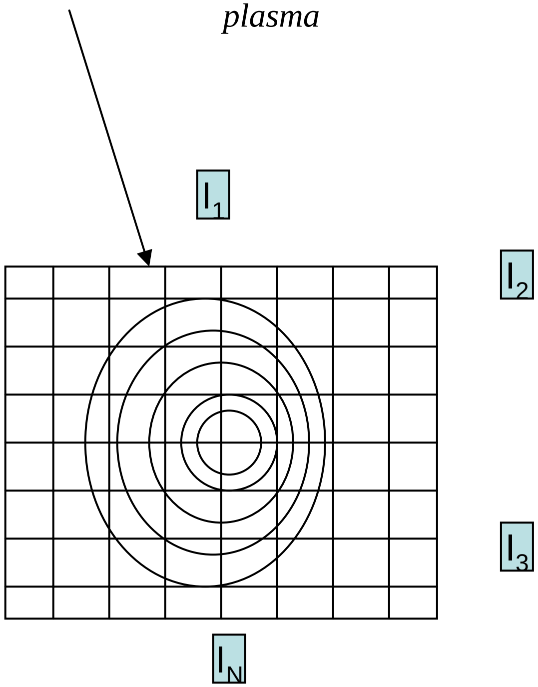
*This is our main interest for SWIM

Coupling to the circuits equation:

Poloidal field coils and metallic structures obey circuits equations coupling them to the plasma:

$$\frac{d}{dt}\left[L_i I_i(t) + \sum_{i\neq j} M_{ij} I_j + \int_{plasma} J(\vec{R}')G(\vec{R}_i,\vec{R}')d\vec{R}'\right] + R_i I_i = V_i, \quad i = 1, N$$

$$\psi_b(\vec{R}) = \int_{plasma} J(\vec{R}')G(\vec{R},\vec{R}')d\vec{R}' + \sum_{i=1}^{N} I_i G(\vec{R},\vec{R}_i)$$



$$G(\mathbf{R};\mathbf{R}') = \frac{\sqrt{RR'}}{2\pi k}\left[(2-k^2)K(k^2) - E(k^2)\right]$$

$$k^2 = \frac{4RR'}{\left[(R+R')^2 + (Z-Z')^2\right]}, \qquad K, E \text{ elliptic integrals}$$

Virtual casing theorem (Green's second identity) is used to turn 2D integral into 1D line integral:

$$\int_{plasma} J(\vec{R}')G(\vec{R},\vec{R}')d\vec{R}' = \oint_{boundary} \frac{d\ell}{R} G(\vec{R},\vec{R}') \frac{\partial U}{\partial n}$$

$$\Delta^* U = \Delta^* \psi, \qquad U = 0 \text{ on boundary}$$

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Alternate Method (used in the TSC code)

$$\rho F(\frac{\partial \vec{V}}{\partial t} + \frac{1}{\tau} V) + \nabla p = \vec{J} \times \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left[\vec{V} \times \vec{B} - \eta \vec{J} \right], \quad \mu_0 \vec{J} = \nabla \times \vec{B}$$

$$p = \frac{\sigma(\Phi)}{V'^{5/3}}$$

F and τ are chosen to speed convergence of relaxation iteration for velocity: results should show convergence as $F \rightarrow 0$.

- Physically, this corresponds to artificially slowing down (critically damping) the Alfven waves and adding an artificial damping term
- Mathematically, it corresponds to a dynamic relaxation iteration method for the equilibrium equation, and just performing one relaxation iteration each time step.
- This method has some advantages in that it has a direct physical analogy. It is also very accurate in calculating Volt-Second consumption, and can easily handle open-field line configurations such as occur during disruptions or during Coaxial Helicity Injection (CHI).

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Extensions

- Include flow in the equilibrium solution
  - – Toroidal flow only
  - – Toroidal + poloidal flow

- Include non-scalar pressure
  - – CGL form $\overleftrightarrow{\mathbf{P}} = p_\perp \overleftrightarrow{I} + (p_\parallel - p_\perp)\dfrac{\vec{B}\vec{B}}{B^2}$
  - – More general form using distribution function

$$\overleftrightarrow{\mathbf{P}}(\vec{R}) = \int d^3\vec{V} f(\vec{R},\vec{V})(\vec{V} - \vec{V}(\vec{R}))(\vec{V} - \vec{V}(\vec{R}))$$